2.6: Applications: Growth and Decay; Mathematics of Finance

Recall the definition of an exponential function:

Definition. An **exponential function** with base a is defined as

$$P(t) = P_0 a^t,$$

where $a > 0$, $a \neq 1$ and P_0 is the quantity present at time $t = 0$

We have **exponential growth** when $a > 1$ and **exponential decay** when $0 < a < 1$.

We use this function when the growth or decay rate is given in "unit of time". For example the annual interest rate is 2% computed one time per year or the population is increasing by 4% per year.

Definition. Let P_0 be the quantity present at time $t = 0$. If the quantity is increasing (or decreasing) at a **continuous growth rate** (or decay), the function is given as

$$P(t) = P_0 e^{kt}.$$

If $k > 0$, we have **exponential growth**.
If $k < 0$, we have **exponential decay**.
The value of k is called the **continuous growth (decay) rate.**

Example. Convert the function $P(t) = 10(1.05)^t$ to the form $P(t) = P_0 e^{kt}$

$P_0 = 10$

$(1.05)^t = e^{kt} \Rightarrow 1.05 = e^k$

$\ln(1.05) = \ln e^k$

$\ln(1.05) = k$

$\Rightarrow 1.05 = e^{\ln(1.05)}$

$\ln(1.05) \approx 0.049$

So we have

$P(t) = 10 \cdot (1.05)^t = 10 \cdot e^{\ln(1.05) \cdot t} = 10 \cdot e^{0.049 \cdot t}$

Example. Convert the function $P(t) = 200e^{-1.7t}$ to the form $P(t) = P_0 a^t$.

$P_0 = 200$

$200 (e^{-1.7})^t$

$e^{-1.7} \approx 0.183$

$= 200 \cdot (0.183)^t$

$a = 0.183$

Which investment is better? 6.0% compounded quartely or 5.9% compounded continuosly?
The **effective rate** will help answer this question.
The effective rate is sometimes called **annual yield**.

$A = P(1+\frac{r}{2})^{2\cdot 1} = P(1+\frac{r}{2})^2$

$A = 1\cdot(1+\frac{0.04}{2})^2 = (1.02)^2$

We have that \$1 at 4% interest (per year) computed semiannually is $1(1.02)^2 = 1.0404$. The increase of 0.040 is 4.04% rather than 4% which is the interest compounded annually. 4% is called the **stated interest rate** and 4.04% is called the **effective interest rate**.

Effective rate:
If r is the annual stated rate of interest, the **effective rate of interest** is:

- 1. $r_E = (1+\frac{r}{m})^m - 1$ when m is the number of compounding periods per year.
- 2. $r_E = e^r - 1$ when compounded continuously.

Example. Lisa decides to invest a \$7,000 bonus check into a savings account. One bank offers 6.0% compounded quartely. Another offers 5.9% compounded continuosly.

1. Which investment will earn the most interest in 5 years?

quartely: $m=4$

$A = P(1+\frac{r}{m})^{t\cdot m}$

$A = 7000(1+\frac{0.06}{4})^{5\cdot 4} \approx 9427.99$

quartely: \$ 9427.99

$\frac{5.9}{100} = 0.059$

Continuosly: $A = Pe^{rt}$

$A = 7000\, e^{0.059\cdot 5} \approx 9401.89$

Continuosly: \$ 9401.89

quartely will earn most interest

2. What is the effective rate in each case?

quartely: $r_E = (1+\frac{r}{m})^m - 1$ $m=4$ $r=0.06$

$r_E = (1+\frac{0.06}{4})^4 - 1 \approx 1.0614 - 1 = 0.0614$

$0.0614 \cdot 100\% = 6.14\%$

Effective rate for quartely: 6.14 %

Continuously: $r_E = e^r - 1$

$= e^{0.059} - 1 = 1.0608 - 1 = 0.0608$

Effective rate: 6.08 %

3. If Lisa chooses the bank with continuous compounding, how long will it take for her $7,000 to grow to a least $10,000.

$r = 0.059$

$$10\,000 = 7000\, e^{0.059 \cdot t}$$

$$\frac{10}{7} = e^{0.059t}$$

$$\ln\left(\frac{10}{7}\right) = \ln e^{0.059t}$$

$$\ln\left(\frac{10}{7}\right) = 0.059t$$

$$t = \frac{\ln\left(\frac{10}{7}\right)}{0.059} \approx 6.05 \text{ years}$$

4. If Lisa chooses the bank with quartely compounding, how long will it take for her $7,000 to grow to a least $10,000.

$r = 0.06 \qquad m = 4$

$$10000 = 7000\left(1 + \frac{0.06}{4}\right)^{4 \cdot t}$$

$$\frac{10}{7} = (1.015)^{4t}$$

$$\ln\left(\frac{10}{7}\right) = \ln(1.015)^{4t}$$

$$\ln\left(\frac{10}{7}\right) = 4t \ln(1.015)$$

$$4t = \frac{\ln\left(\frac{10}{7}\right)}{\ln(1.015)} \Rightarrow t = \frac{\ln\left(\frac{10}{7}\right)}{4 \cdot \ln(1.015)}$$

$$\approx 5.99$$

It takes 6 years

Present Value:
P is the amount that should be deposited today to produce A dollars in t years. The value of P is called the **present value** of A dollars.

Example. Curt must make a balloon payment of \$15,000 in 3 years. Find the present value of the payment if it includes annual interest of 7% compounded monthly.

$$A = P\left(1+\frac{r}{m}\right)^{m\cdot t}$$

$A = 15000$
$m = 12$
$r = 0.07$
$t = 3$
P unknown

$$15000 = P\left(1+\frac{0.07}{12}\right)^{12\cdot 3}$$

$$15000 = P \cdot 1.233$$

$$P = \frac{15000}{1.233} \approx 12165.5$$

The present value is: \$ 12165.5

Limited Growth Functions

Example. Sales of a new model of digital camera are approximated by

$$S(x) = 4000 - 3000e^{-x},$$

where x represents the number of years that the digital camera has been on the market, and $S(x)$ represents sales in thousands of dollars.

1. Find the sales in year 0.

$S(0) = 4000 - 3000e^{-0} = 1000$

Sales are \$ 1000 000 in year 0.

2. When will sales reach \$3,500.000?

$3500 = 4000 - 3000e^{-x}$

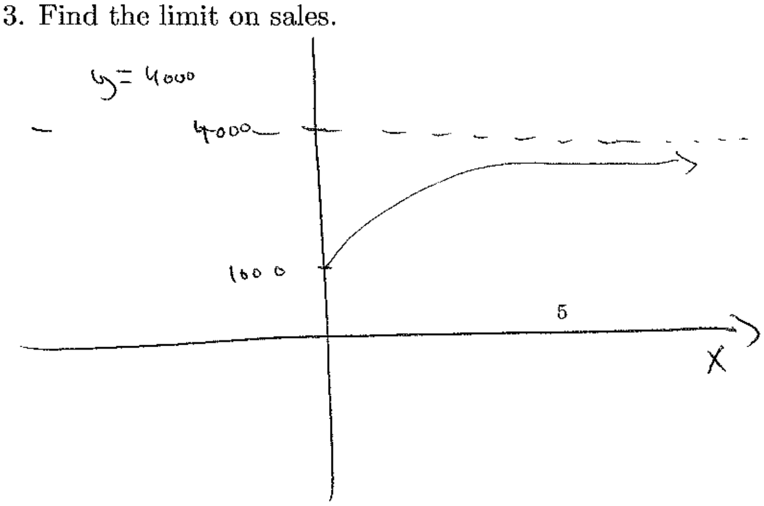
$3000e^{-x} = 500$

$e^{-x} = \frac{500}{3000}$

$x = -\ln\left(\frac{5}{30}\right) = -\ln\left(\frac{1}{6}\right) \approx 1.79$

It will reach \$ 3,500.000 after 1.79 years

3. Find the limit on sales.



As x gets large, e^{-x} goes to zero.

So $S(x) = 4000 - 3000e^{-x}$ goes to 4000

Limiting sales are 4000,000