2.5: Logarithmic Functions

Logarithm:
For $a > 0$ and $a \neq 1$, and $x > 0$

$$y = \log_a(x) \text{ means } a^y = x.$$

$\log_{10} 100 = 2$ means $10^2 = 100$.
$\log_2 8 = 3$ means $2^3 = 8$.
$\log_3 1 = 0$ means $3^0 = 1$

Example. Evaluate:

1. $\log_2 16 = \log_2 2^4 = 4$ since $2^4 = 16$

2. $\log_3 \frac{1}{81} = \log_3\left(\frac{1}{3^4}\right) = \log_3\left(3^{-4}\right) = -4$ since $3^{-4} = \frac{1}{81}$

Definition. For $a > 0$ and $a \neq 1$, the **logarithmic function** of base a is defined as

$$f(x) = \log_a(x)$$

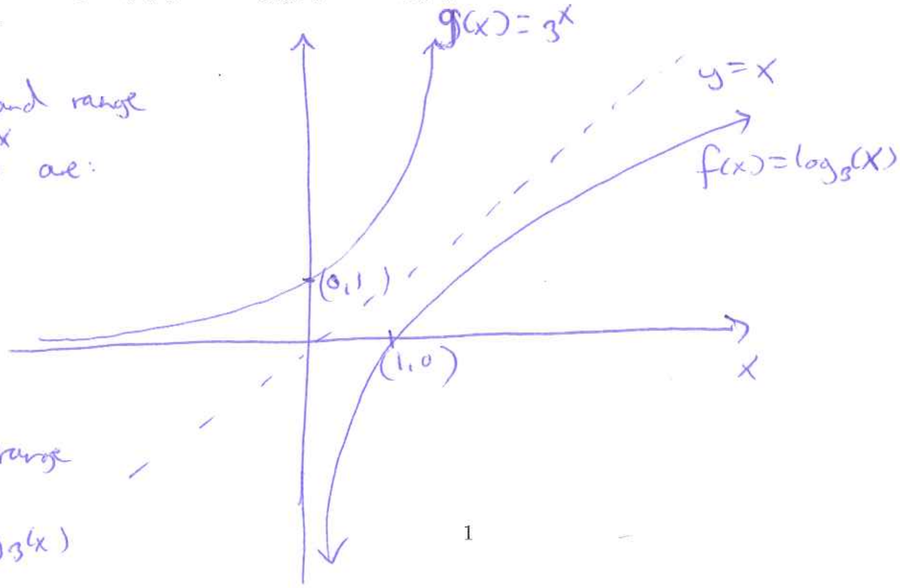
for $x > 0$.

Example. Graph $f(x) = \log_3(x)$ and $g(x) = 3^x$. Find the domain and range of both functions.

The domain and range of $g(x) = 3^x$ are:

$D = (-\infty, \infty)$

$R = (0, \infty)$



The domain and range of $f(x) = \log_3(x)$ are:

$D = (0, \infty)$

$R = (-\infty, \infty)$

$f(x) = \log_a(x)$ has a vertical asymptote: $X = 0$

$a > 0$, $a \neq 1$

The domain of $f(x) = \log_a(x)$ is $D = (0, \infty)$ and the range is $R = (-\infty, \infty)$.

For any number x, if $f(x) = y$, then $g(y) = x$, we say that f and g are **inverse functions** of each other.

$f(x) = \log_2(x)$ and $g(x) = 2^x$ are inverse functions.

For $a > 0$ and $a \neq 1$, $f(x) = \log_a(x)$ and $g(x) = a^x$ are inverse functions.

We can graph the inverse of f by reflecting the graph of f about the line $y = x$.

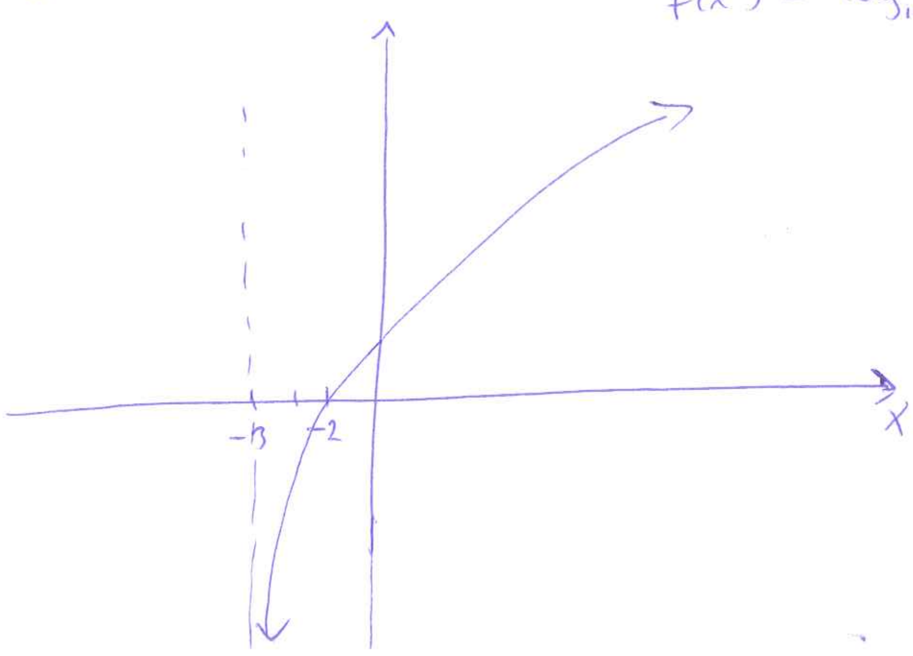
Example. Find the domain of $f(x) = \log_{10}(x + 3)$.

$X + 3 > 0$

$X > -3$

$D = (-3, \infty)$

$f(x) = \log_{10}(X+3)$



Properties of Logarithms: Let x, y be an positive real numbers and let r be any real number. Let a be a positive real number, $a \neq 1$. Then

- 1. $\log_a xy = \log_a x + \log_a y$
- 2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
- 3. $\log_a x^r = r \log_a x$
- 4. $\log_a a = 1$
- 5. $\log_a 1 = 0$
- 6. $\log_a a^r = r$
- 7. $a^{\log_a x} = x$.

Example. Write as a common logarithm:

1. $\log_2(x+1) + \log_2(x-1)$ $= \log_2[(x+1)(x-1)] = \log_2(x^2-1)$

by property 1.

2. $2\log_3(z+2) - \log_3(z+3)$ $\overset{\text{Property 3}}{=} \log_3(z+2)^2 - \log(z+3)$

$\overset{\text{Property 2}}{=} \log_3\left[\frac{(z+2)^2}{(z+3)}\right]$

Example. Expand the logarithm

$$\log_3\left(\frac{x^2-4}{xy}\right)^2$$

$= 2\log_3\left(\frac{(x+2)(x-2)}{xy}\right) = 2\left[\log_3[(x+2)(x-2)] - \log(xy)\right]$

$= 2\left[\log_3(x+2) + \log_3(x-2) - \log(x) - \log(y)\right]$

$\log_{10} x$ is abbreviated $\log(x)$

$\log_a x$ is abbreviated $\ln x$.
$\ln x$ is called the **natural logarithm.**

In order to graph a logarithmic function on your calculator for a base other than e or 10, the following theorem is useful:

The change-of-base Theorem for logarithms:
Let x be any positive real number and let a and b positive real numbers, $a \neq 1$, $b \neq 1$, then

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

Using $\ln x$ for $\log_e x$ gives the special case:

$$\log_a x = \frac{\ln x}{\ln a}.$$

Example. Evaluate: $\log_7 90$ $= \dfrac{\log_{10} 90}{\log_{10} 7} = \dfrac{\log(90)}{\log 7} \approx \underline{\underline{2.31}}$

or $\log_7 90 = \dfrac{\ln(90)}{\ln(7)} \approx \underline{\underline{2.31}}$

Solving Logarithmic Equations:

Example. Solve the following Logarithmic Equations:

1. $\log_2 x = 3$ or $\log_2 X = 3$ means $X = 2^3 = \underline{\underline{8}}$

By property 7

$2^{\log_2 X} = 2^3$

$\underline{\underline{X = 8}}$

2. $\log_3(4x-1) = 2$

$$3^{\log_3(4x-1)} = 3^2$$

$$4x-1 = 9$$

$$4x = 10 \qquad x = \frac{10}{4} = \frac{5}{2}$$

plug in $x = \frac{5}{2}$ into,

$\log_3(4x-1)$,

$$4x-1 = \frac{5}{2}\cdot 4 - 1 = 9 > 0$$

Solution $\boxed{\frac{5}{2}}$

3. $\log(x-1) - \log(x+2) = 1$

Property 2 $\quad \log\left(\frac{x-1}{x+2}\right) = 1$

$$10^{\log\left(\frac{x-1}{x+2}\right)} = 10^1$$

Property 7 with $a=10$ $\quad \frac{x-1}{x+2} = 10$

$$x-1 = 10(x+2)$$

$$x-1 = 10x + 20$$

$$9x = -21$$

$x = -\frac{21}{9}$ Cannot be a Solution.

plug $x = -\frac{21}{9}$ into

$\log(x-1)$,

we get

$$\log\left(-\frac{21}{9} - 1\right) = \log\left(-\frac{31}{9}\right)$$

not defined

no Solution

4. $\ln(x-3) + \ln(x+3) = \ln(x)$

Property 1: $\quad \ln[(x-3)(x+3)] = \ln(x)$

$$e^{\ln[(x-3)(x+3)]} = e^{\ln(x)}$$

Property 7 $\quad (x-3)(x+3) = x$

$$x^2 - 9 = x$$

$$x^2 - x - 9 = 0$$

$$x = \frac{1 \pm \sqrt{1 - 4\cdot 1\cdot(-9)}}{2\cdot 1} = \frac{1 \pm \sqrt{37}}{2} \approx 3.54 \text{ or } -2.54$$

plug in $x = -2.54$ into $\ln(x-3)$ we get $\ln(-2.54-3)$ not defined

Plug in $X = 3.54$

into $\ln(x-3)$

we get $\ln(3.54-3) = \ln(0.54)$

OK.

Plug in $X = 3.54$ into

$\ln(X+3)$

we get $\ln(3.54+3) = \ln(6.54)$

ok.

Solution is $\underline{\underline{3.54}}$

Solving Exponential Equations:

Example. Solve the following Exponential Equations:

1. $2^x = 3$

$\ln 2^x = \ln 3$

$x \ln 2 = \ln 3$

$x = \frac{\ln 3}{\ln 2} = \log_2 3$

2. $e^{3x} = 2$

$\ln e^{3x} = \ln 2$

$3x = \ln 2$

$x = \frac{\ln(2)}{3}$

3. $3^{x+2} = 5^{2x}$

$\ln(3^{x+2}) = \ln(5^{2x})$

Property 3 $(x+2)\ln(3) = 2x(\ln 5)$

$x\ln 3 + 2\ln 3 = 2x\ln 5$

$2\ln 3 = 2x\ln 5 - x\ln 3$

$2\ln 3 = x(2\ln 5 - \ln 3)$

$\frac{2\ln 3}{2\ln 5 - \ln 3} = x$

$\log_3 3^{(x+2)} = \log_3 5^{2x}$

Property 6

$x+2 = \log_3 5^{2x}$

$x+2 = 2x \log_3 5$

$2 = 2x\log_3 5 - x$

$2 = x(2\log_3 5 - 1)$

$\frac{2}{2\log_3 5 - 1} = x$

$\log_3 5 = \frac{\log 5}{\log 3}$

Example. With an inflation rate of 3% per year, how long will it take for prices to double:

Recall $\boxed{A = P\left(1+\frac{r}{m}\right)^{t\cdot m}}$ $m=1$

$$A = P(1+r)^t$$

Initial value is P $r = 0.03$

Double: $2P$

$$2\cancel{P} = \cancel{P}(1+0.03)^t$$

$$2 = (1+0.03)^t$$

$$2 = (1.03)^t$$

$$\ln(2) = \ln\left((1.03)^t\right)$$

$$\ln(2) = t \cdot \ln(1.03)$$

$$t = \frac{\ln(2)}{\ln(1.03)} \approx 23.4 \text{ years}$$

It will take 23.4 years for prices to double.