2.5: Logarithmic Functions

Logarithm:
For $a > 0$ and $a \neq 1$, and $x > 0$

$$y = \log_a(x) \text{ means } a^y = x.$$

$\log_{10} 100 = 2$ means $10^2 = 100$.
$\log_2 8 = 3$ means $2^3 = 8$.
$\log_3 1 = 0$ means $3^0 = 1$

Example. Evaluate:

1. $\log_2 16$

2. $\log_3 \frac{1}{81}$

Definition. For $a > 0$ and $a \neq 1$, the **logarithmic function** of base a is defined as

$$f(x) = \log_a(x)$$

for $x > 0$.

Example. Graph $f(x) = \log_3(x)$ and $g(x) = 3^x$. Find the domain and range of both functions.

The domain of $f(x) = \log_a(x)$ is $D = (0, \infty)$ and the range is $R = (-\infty, \infty)$.

For any number x, if $f(x) = y$, then $g(y) = x$, we say that f and g are **inverse functions** of each other.

$f(x) = \log_2(x)$ and $g(x) = 2^x$ are inverse functions.

For $a > 0$ and $a \neq 1$, $f(x) = \log_a(x)$ and $g(x) = a^x$ are inverse functions.

We can graph the inverse of f by reflecting the graph of f about the line $y = x$.

Example. Find the domain of $f(x) = \log_{10}(x + 3)$.

Properties of Logarithms: Let x, y be an positive real numbers and let r be any real number. Let a be a positive real number, $a \neq 1$. Then

- 1. $\log_a xy = \log_a x + \log_a y$
- 2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
- 3. $\log_a x^r = r \log_a x$
- 4. $\log_a a = 1$
- 5. $\log_a 1 = 0$
- 6. $\log_a a^r = r$
- 7. $a^{\log_a x} = x$.

Example. Write as a common logarithm:

1. $\log_2(x+1) + \log_2(x-1)$

2. $2\log_3(z+2) - \log_3(z+3)$

Example. Expand the logarithm

$$\log_3 \left(\frac{x^2 - 4}{xy} \right)^2$$

$\log_{10} x$ is abbreviated $\log(x)$

$\log_a x$ is abbreviated $\ln x$.
$\ln x$ is called the **natural logarithm.**

In order to graph a logarithmic function on your calculator for a base other than e or 10, the following theorem is useful:

The change-of-base Theorem for logarithms:
Let x be any positive real number and let a and b positive real numbers, $a \neq 1$, $b \neq 1$, then

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

Using $\ln x$ for $\log_e x$ gives the special case:

$$\log_a x = \frac{\ln x}{\ln a}.$$

Example. Evaluate: $\log_7 90$

Solving Logarithmic Equations:

Example. Solve the following Logarithmic Equations:

1. $\log_2 x = 3$

2. $\log_3(4x - 1) = 2$

3. $\log(x - 1) - \log(x + 2) = 1$

4. $\ln(x - 3) + \ln(x + 3) = \ln(x)$

Solving Exponential Equations:

Example. Solve the following Exponential Equations:

1. $2^x = 3$

2. $e^{3x} = 2$

3. $3^{x+2} = 5^{2x}$

Example. With an inflation rate of 3% per year, how long will it take for prices to double: