2.3 Polynomial and rational functions

Definition. A **Polynomial function** is defined by

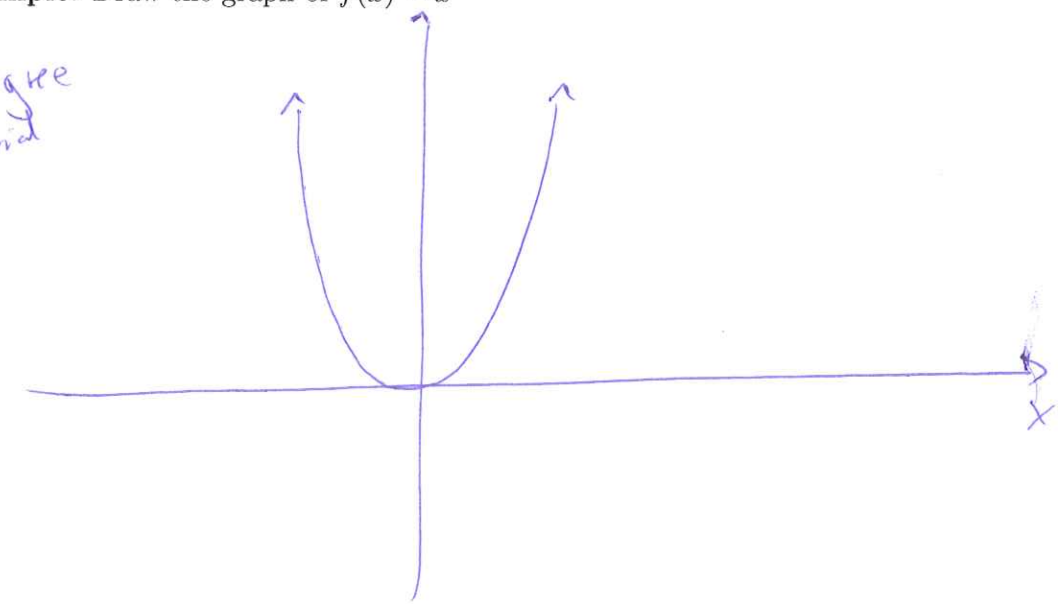
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0,$$

where $a_n, a_{n-1}, ..., a_1, a_0$ are real numbers.
a_n is called the **leading coefficient**. The degree of the polynomial is n.

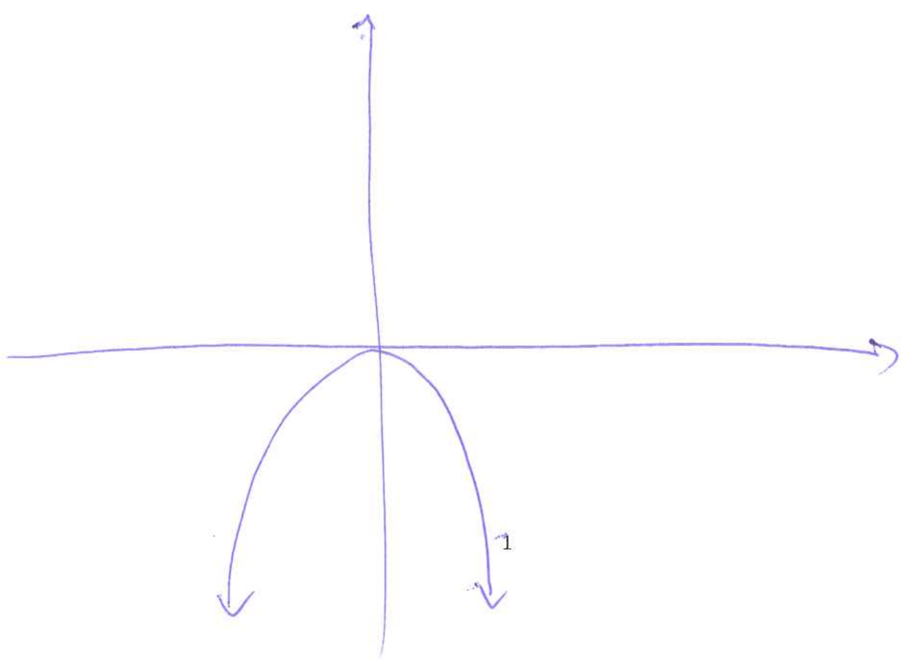
When $n = 2$, we have a quadratic function, $f(x) = a_2x^2 + a_1x + a_0$ and when $n = 1$, we have a linear function, $f(x) = a_1x + a_0$.

Example. Draw the graph of $f(x) = x^2$

Even degree polynomial

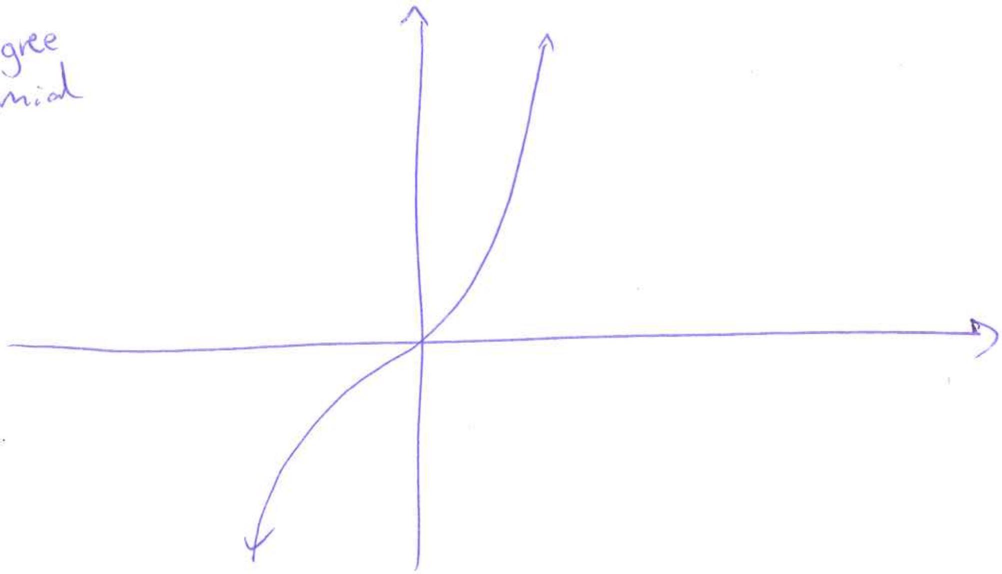


Example. Draw the graph of $f(x) = -x^2$

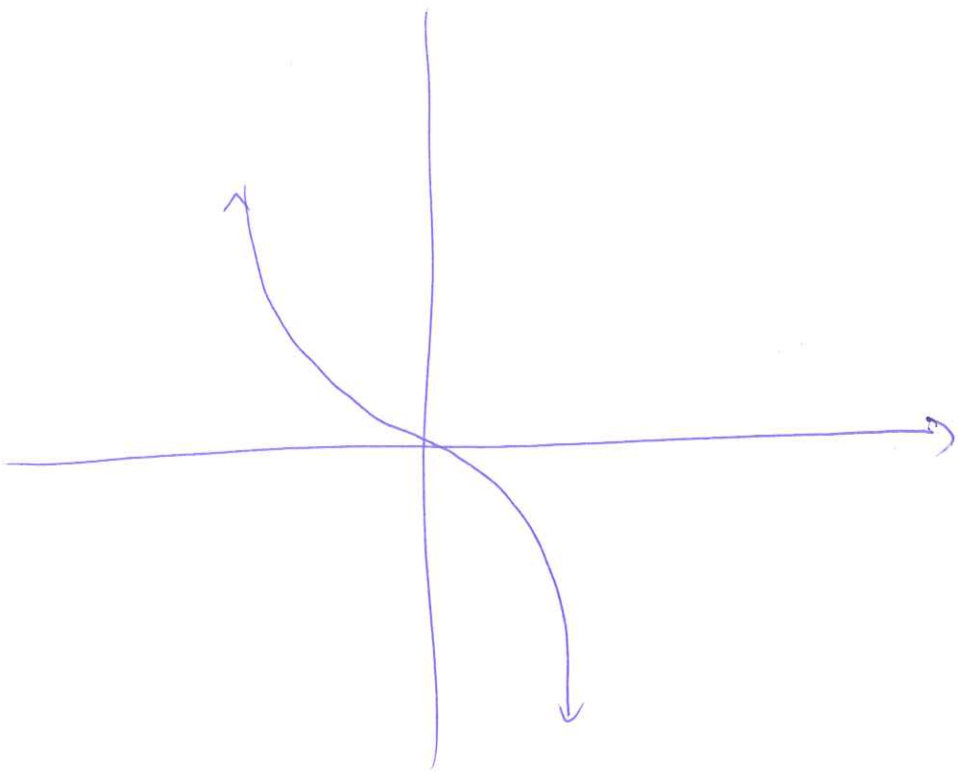


Example. Draw the graph of $f(x) = x^3$

Odd degree polynomial



Example. Draw the graph of $f(x) = -x^3$



Properties of polynomial functions

- A polynomial function of degree n can have at most $n-1$ turning points. Conversely, if the graph of a polynomial function has $n-1$ turning points, then it must have degree at least n.
- For a polynomial function of **even degree** and with **positive** leading coefficient, both ends of its graph go **up**.
- For a polynomial function of **even degree** and with **negative** leading coefficient, both ends of its graph go **down**.
- For a polynomial function of **odd degree** and with **positive** leading coefficient, the left end of its graph goes **down** and the right end goes **up**.
- For a polynomial function of **odd degree** and with **negative** leading coefficient, the left end of its graph goes **up** and the right end goes **down**.

Example. Draw the graph of $f(x) = x^3 - 5x^2 + 2x + 8$ and determine the x-intercepts.

$f(x) = (x+1)(x-2)(x-4)$

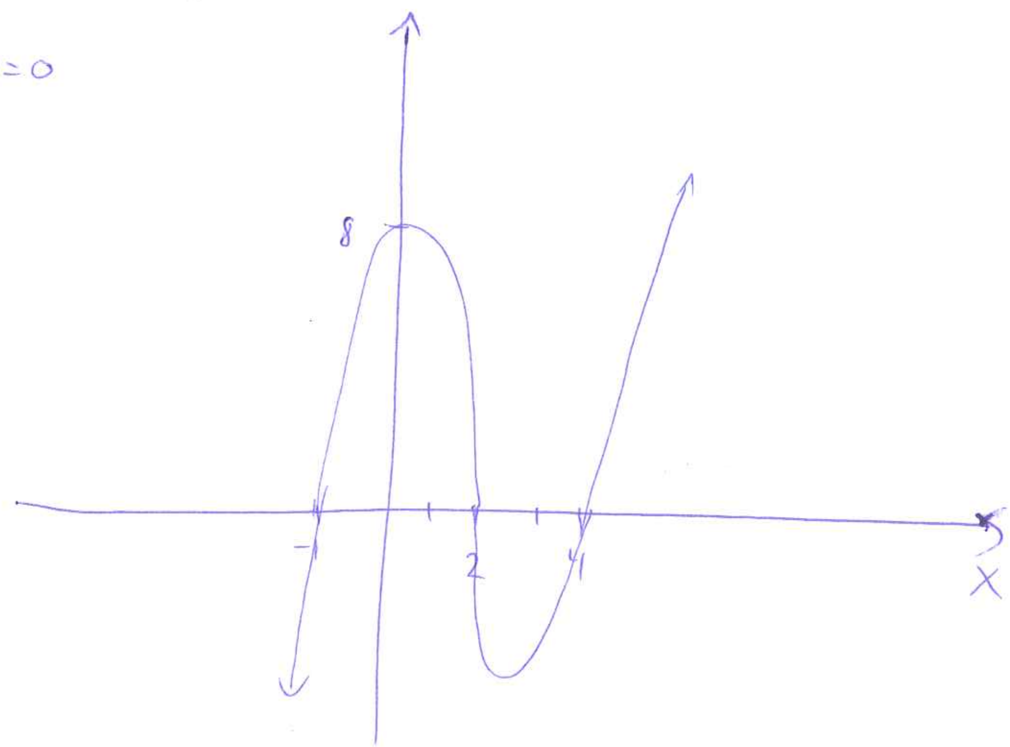
x-intercept: $f(x) = 0$

$x = -1$

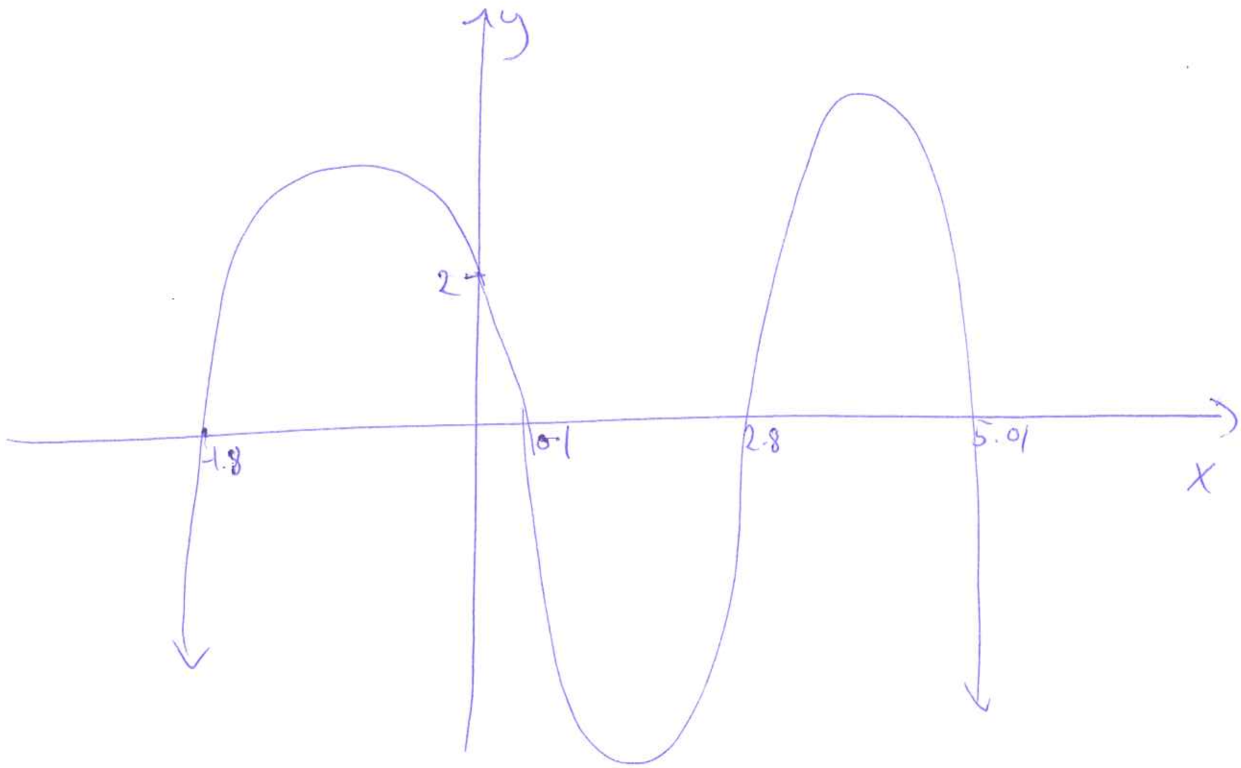
$x = 2$

$x = 4$

2 turning points



Example. Use your calculator to draw the graph of $f(x) = -2x^4 + 12x^3 - 50x + 2$.



Window:

$[-2, 5.5] \times [-50, 70]$

Definition. A rational function is defined by
$$f(x) = \frac{p(x)}{q(x)},$$
where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

Example. Draw the graph of $f(x) = \frac{1}{x}$ and

fill out the tables

Small $|X|$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1	
$y = \frac{1}{x}$	-10	-100	-1000	1000	100	10	

and

Large $|X|$

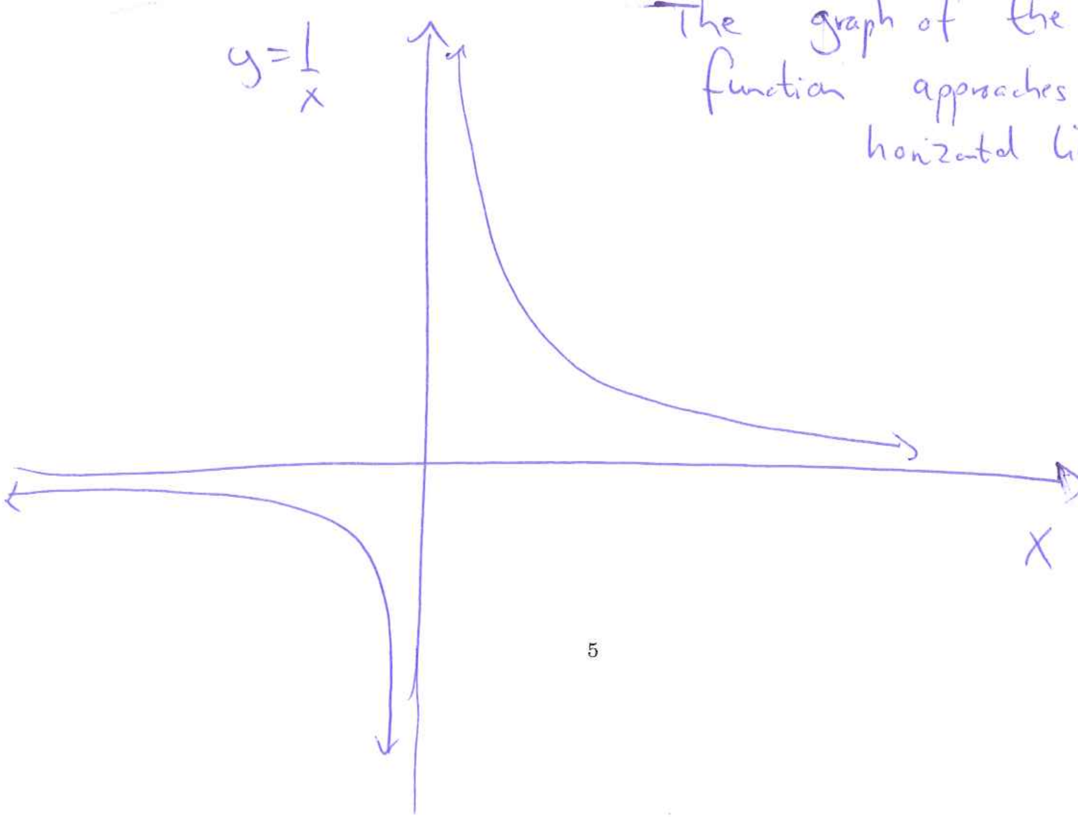
x	-1000	-100	-10	-1	1	10	100	1000
$y = \frac{1}{x}$	-0.001	-0.01	-0.1	-1	1	0.1	0.01	0.001

As $|X|$ gets ~~smaller~~ closer and ~~smaller~~ closer to zero, $|y| = \left|\frac{1}{X}\right|$ gets larger and larger. Thus, the graph of the function approaches the vertical line $X = 0$.

As $|X|$ gets larger and larger, $y = \frac{1}{X}$ gets closer and closer to 0.

$y = \frac{1}{X}$

The graph of the function approaches the horizontal line $y = 0$.



Result 0.1. *If a function gets larger and larger in magnitude wihout bound as x approaches the number k, then the line $x = k$ is a* ***vertical asymptote****.*
If the values of y approach a number k as $|x|$ gets larger and larger, the line $y = k$ is a ***horizontal asymptote****.*

Example. Find the asymptotes and graph the function

$$f(x) = \frac{2x^2 + x + 1}{x^2 - 25}$$

Vertical asymptotes:

$x^2 - 25 = 0$

$x^2 = 25$

$x = \pm 5$

Horizontal asymptotes:

Let X gets large or small.

Then $2x^2 + x + 1 \approx 2x^2$ and $x^2 - 25 \approx x^2$,

So $\frac{2x^2+x+1}{x^2-25} \approx \frac{2x^2}{x^2} = 2$.

i.e for large $|x|$,

$f(x)$ behaves like $y = 2$,

So $y = 2$ is a horizontal asymptote

y-intercept:

$f(0) = -\frac{1}{25}$

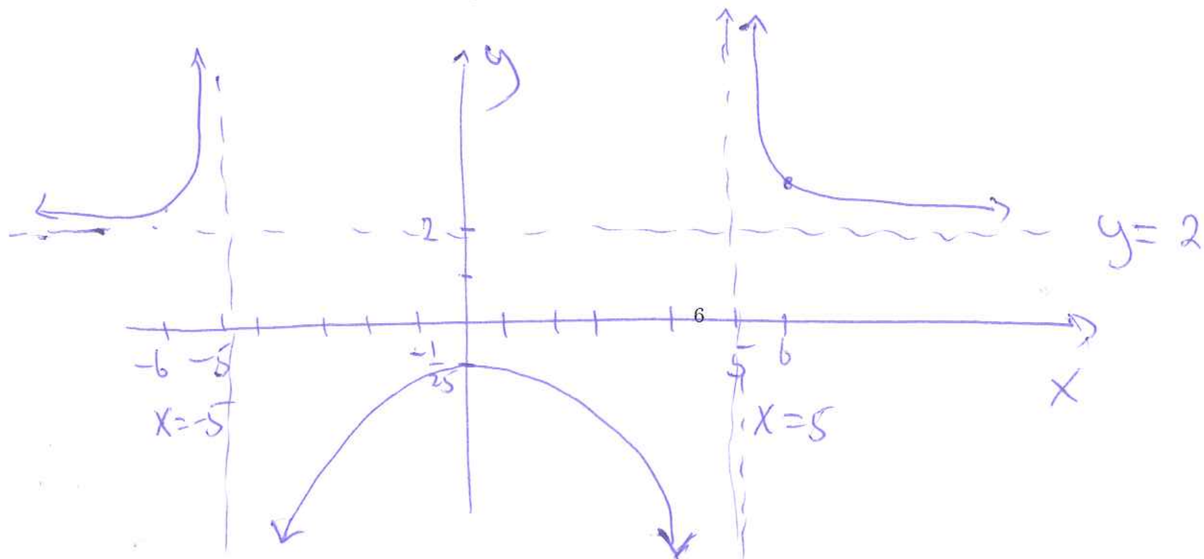
Plug in $x = 6$

$f(6) \approx 7.2 > 2$

So the graph is above the line $y = 2$ for $x > 5$

Plug in $x = -6$,

$f(-6) \approx 6.1 > 2$, So the graph is above the line $y = 2$ for $x < 5$.



Example. Find the asymptotes and graph the function

$$g(x) = \frac{x-4}{x^2+x-2}$$

$x^2 + x - 2 = 0$

$(x-1)(x+2) = 0$

$\underline{x=1}$ $\underline{x=-2}$: vertical asymptotes:

As $|x|$ gets larger,

$x - 4 \approx x$

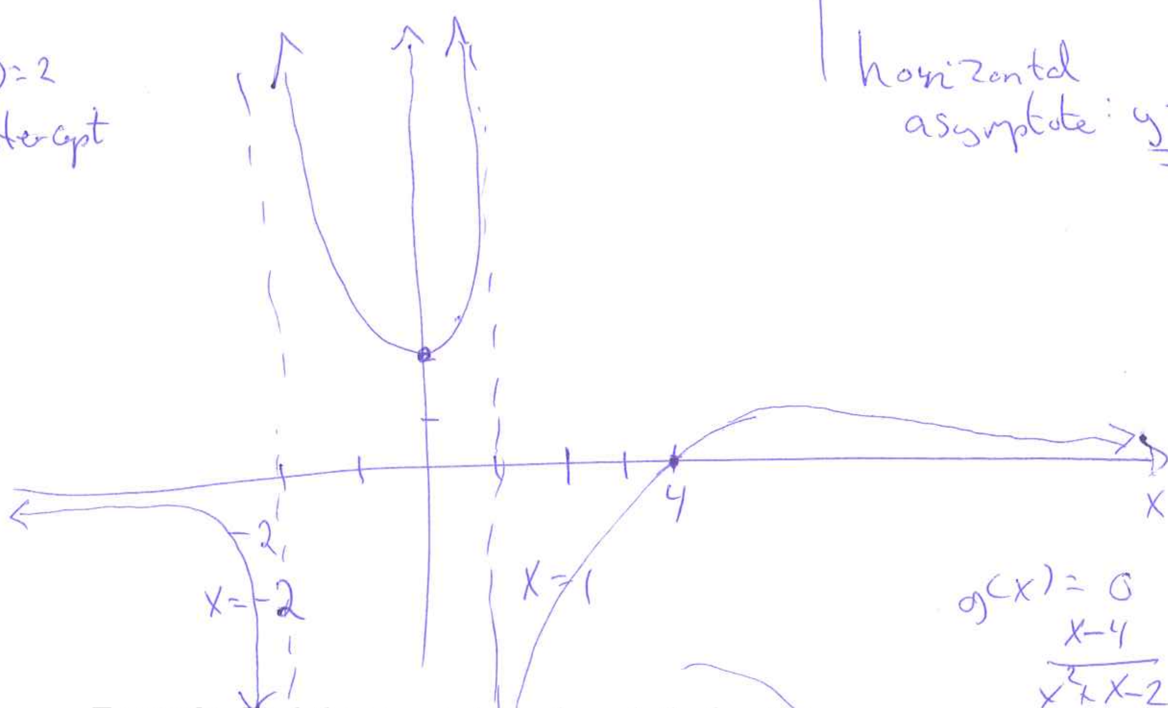
$x^2 + x - 2 \approx x^2$

$g(x) = \frac{x-4}{x^2+x-2} \approx \frac{x}{x^2} = \frac{1}{x} \to 0$

horizontal asymptote: $\underline{y=0}$

$g(0) = 2$

y-intercept



$g(x) = 0$

$\frac{x-4}{x^2+x-2} = 0 \quad (x^2+x-2)$

$x - 4 = 0$

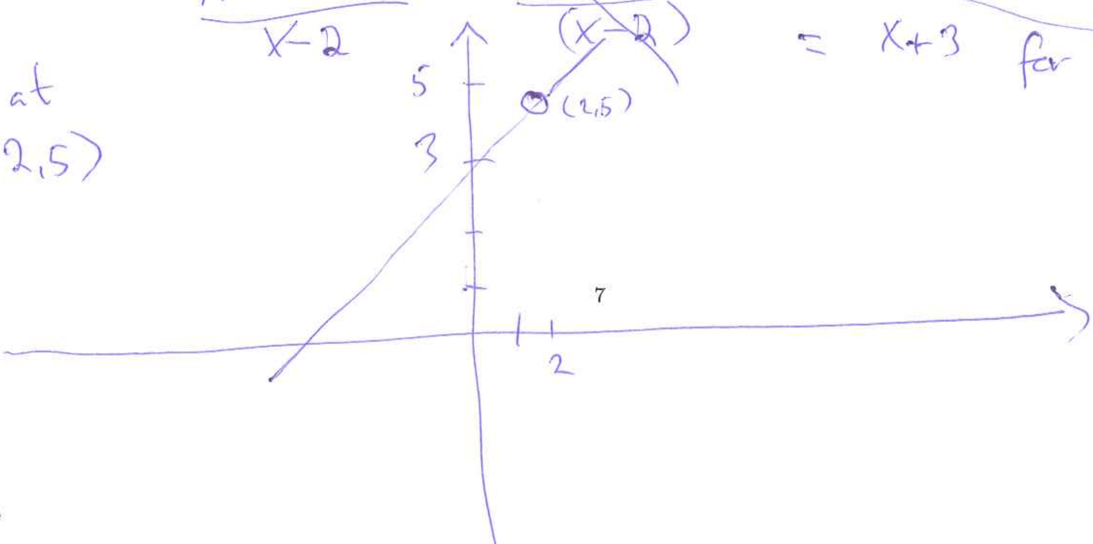
$x = 4$ - x-intercept

Example. Find the asymptotes and graph the function

$$h(x) = \frac{x^2+x-6}{x-2}$$

$\frac{x^2+x-6}{x-2} = \frac{\cancel{(x-2)}(x+3)}{\cancel{(x-2)}} = x+3$ for $x \neq 2$

Hole at $(2,5)$



Example. Suppose a cost-benefit model is given by

$$y = g(x) = \frac{6x}{101 - x},$$

where y is the cost in thousands of dollars of removing x percent of a certain pollutant.

1. Find the cost of removing each percent of pollutants: 0%, 50%, 80%, 90%, 95%, 99%, 100%.

$g(0) = \frac{6 \cdot 0}{101 - 0} = 0$ $\quad g(50) = \frac{6 \cdot 50}{101 - 50} = 5.9$ or \$5 900

$g(80) = \frac{6 \cdot 80}{101 - 80} = 22.9$ or \$22900 $\quad g(90) = 49.1$ or \$49100

$g(95) = 95$ or \$95000 $\quad$ $g(99) = 297$ or \$297 000

$g(100) = 600$ or \$600 000

2. Graph the function.