2.3 Polynomial and rational functions

Definition. A **Polynomial function** is defined by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0,$$

where $a_n, a_{n-1}, ..., a_1, a_0$ are real numbers.
a_n is called the **leading coefficient**. The degree of the polynomial is n.

When $n = 2$, we have a quadratic function, $f(x) = a_2x^2 + a_1x + a_0$ and when $n = 1$, we have a linear function, $f(x) = a_1x + a_0$.

Example. Draw the graph of $f(x) = x^2$

Example. Draw the graph of $f(x) = -x^2$

Example. Draw the graph of $f(x) = x^3$

Example. Draw the graph of $f(x) = -x^3$

Properties of polynomial functions

- A polynomial function of degree n can have at most $n-1$ turning points. Conversely, if the graph of a polynomial function has $n-1$ turning points, then it must have degree at least n.
- For a polynomial function of **even degree** and with **positive** leading coefficient, both ends of its graph go **up**.
- For a polynomial function of **even degree** and with **negative** leading coefficient, both ends of its graph go **down**.
- For a polynomial function of **odd degree** and with **positive** leading coefficient, the left end of its graph goes **down** and the right end goes **up**.
- For a polynomial function of **odd degree** and with **negative** leading coefficient, the left end of its graph goes **up** and the right end goes **down**.

Example. Draw the graph of $f(x) = x^3 - 5x^2 + 2x + 8$ and determine the x-intercepts.

Example. Use your calculator to draw the graph of $f(x) = -2x^4 + 12x^3 - 50x + 2$.

Definition. A rational function is defined by

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

Example. Draw the graph of $f(x) = \frac{1}{x}$ and

fill out the tables

x	-0.1	-0.01	-0.001	0.001	0.01	0.1	
$y = \frac{1}{x}$							

and

x	-1000	-100	-10	-1	1	10	100	1000
$y = \frac{1}{x}$								

Result. If a function gets larger and larger in magnitude wihout bound as x approaches the number k, then the line $x = k$ is a **vertical asymptote**.
If the values of y approach a number k as $|x|$ gets larger and larger, the line $y = k$ is a **horizontal asymptote**.

Example. Find the asymptotes and graph the function

$$f(x) = \frac{2x^2 + x + 1}{x^2 - 25}$$

Example. Find the asymptotes and graph the function

$$g(x) = \frac{x - 4}{x^2 + x - 2}$$

Example. Find the asymptotes and graph the function

$$h(x) = \frac{x^2 + x - 6}{x - 2}$$

Example. Suppose a cost-benefit model is given by

$$y = g(x) = \frac{6x}{101 - x},$$

where y is the cost in thousands of dollars of removing x percent of a certain pollutant.

1. Find the cost of removing each percent of pollutants: $0\%, 50\%, 80\%, 90\%, 95\%, 99\%, 100\%$.

2. Graph the function.