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On the Spectrum of the Dirichlet Laplacian in a Narrow Infinite Strip

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To Mikhail Shlëmovich Birman on his 80th birthday

Abstract. This is a continuation of the paper [**3**]. We consider the Dirichlet Laplacian in a family of unbounded domains $\{x \in \mathbb{R},\ 0 < y < \epsilon h(x)\}$. The main assumption is that $x = 0$ is the only point of global maximum of the positive, continuous function $h(x)$. We show that the number of eigenvalues lying below the essential spectrum indefinitely grows as $\epsilon \to 0$, and find the two-term asymptotics in $\epsilon \to 0$ of each eigenvalue and the one-term asymptotics of the corresponding eigenfunction. The asymptotic formulae obtained involve the eigenvalues and eigenfunctions of an auxiliary ODE on $\mathbb{R}$ that depends only on the behavior of $h(x)$ as $x \to 0$.

The proof is based on a detailed study of the resolvent of the operator Δ_ϵ.

1. Introduction

This paper is a continuation of the authors' work [**3**], where we studied the spectrum of the Dirichlet Laplacian $\Delta_{\epsilon,D}$ in a narrow strip

$$\Omega_\epsilon = \{(x, y) : x \in I,\ 0 < y < \epsilon h(x)\}.$$

The main objective in [**3**] was to understand the behavior of the eigenvalues as $\epsilon \to 0$, and the main assumption was that I is a finite segment and the continuous function $h(x)$ has on I a single point of global maximum. We found the two-term asymptotics (in $\epsilon \to 0$) of each eigenvalue, and also the one-term asymptotics of each eigenfunction. Our approach was based upon a careful study of the resolvent

$$\mathfrak{A}_{\epsilon,D}^{-1} := (\Delta_{\epsilon,D} - K\epsilon^{-2})^{-1}$$

with an appropriate choice of the constant K; see (2.6) below. Note that we consider the Laplacian as a positive operator, so that

$$\Delta\psi = -\psi''_{xx} - \psi''_{yy}.$$

Here we apply the same approach to two other problems of a similar nature. One of them concerns the case $I = \mathbb{R}$, and the assumptions about $h(x)$ are basically the same as in [**3**], complemented by a mild additional condition as $|x| \to \infty$. In

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another problem I is a finite segment (as it was in [**3**]), but the Dirichlet condition at the vertical parts of $\partial\Omega_\epsilon$ is replaced by the Neumann condition. This gives rise to the operator $\Delta_{\epsilon,DN}$ that was not discussed in [**3**]. In both cases, and especially in the second one, the difference with the original problem studied in [**3**] looks minor. However, some technical tools used there no more apply in the new situation, and one has to look for appropriate substitutes.

Note that the case of the Neumann boundary condition on the whole of $\partial\Omega_\epsilon$ is simpler than the case of the Dirichlet condition; see its analysis in [**7**] and [**6**]. The results in [**7**], [**6**] concern a much wider class of domains than those in our paper [**3**].

We believe that our results for unbounded Ω_ϵ, i.e. for $I = \mathbb{R}$, are of some independent interest. They easily extend to the case when I is a half-line; we leave it to the reader. The results for the operator $\Delta_{\epsilon,N}$ are of a more technical character. They are useful, since they allow one to apply the Dirichlet–Neumann bracketing for study of some other problems. In particular, in our problem for $I = \mathbb{R}$ this gives a simple way to obtain the asymptotics of eigenvalues, avoiding an analysis of the resolvent; we show this in Section 6. One more problem concerns the case when $I = \mathbb{R}$ and the function $h(x)$ is periodic (Laplacian in a thin periodic waveguide). Here the spectrum of $\Delta_{\epsilon,D}$ has the band-gap structure, and we study the location of bands and give a bound for their widths; it turns out that they decay exponentially as $\epsilon \to 0$. We present our corresponding results in a separate paper [**4**].

We complement these comments at the end of Section 2, after introducing the necessary notation and formulating some of our main results.

2. The case of a finite segment: Setting of the problem and formulation of main results

2.1. Preliminaries. Let $I = [-a, b]$ be a finite segment and $h(x) > 0$ be a continuous function on I. We assume that

(i) $x = 0$ is the only point of global maximum of $h(x)$ on I.

(ii) The function $h(x)$ is C^1 on $I \setminus \{0\}$, and in a neighborhood of $x = 0$ it admits an expansion

$$h(x) = \begin{cases} M - c_+ x^m + O\big(x^{m+1}\big), & x > 0, \\ M - c_-|x|^m + O\big(|x|^{m+1}\big), & x < 0, \end{cases} \tag{2.1}$$

where $M, c_\pm > 0$ and $m \geq 1$.

We consider the Laplacian in Ω_ϵ, and we always impose the boundary conditions

$$\psi(x, 0) = \psi(x, \epsilon h(x)) = 0. \tag{2.2}$$

The conditions at $x = -a$ and $x = b$ can be either Dirichlet or Neumann, and we denote the corresponding operators as $\Delta_{\epsilon,D}$ and $\Delta_{\epsilon,DN}$, respectively; in [**3**] the operator $\Delta_{\epsilon,D}$ was denoted as Δ_ϵ. For the sake of brevity, sometimes we speak about the 'D-problem' and the 'DN-problem'.

Denote

$$H^{1,d}(\Omega_\epsilon) = \{\psi \in H^1(\Omega_\epsilon) : \ \psi(x, 0) = \psi(x, \epsilon h(x)) = 0\}.$$

The conditions (2.2) imply

$$\int_0^{\epsilon h(x)} \psi_y'(x,y)^2 dy \geq \frac{\pi^2}{\epsilon^2 h^2(x)} \int_0^{\epsilon h(x)} \psi^2(x,y)dy; \tag{2.3}$$

here ψ is a smooth real-valued function. The nature of the problem allows us to work with such functions only. By (2.3),

$$\int_{\Omega_\epsilon} |\nabla\psi|^2 dxdy \geq \frac{\pi^2}{M^2\epsilon^2} \int_{\Omega_\epsilon} \psi^2 dxdy, \qquad \forall\psi \in H^{1,d}(\Omega_\epsilon). \tag{2.4}$$

It is convenient for us to work with the quadratic form

$$\mathfrak{a}_\epsilon[\psi] = \int_{\Omega_\epsilon} \left(|\nabla\psi|^2 - \frac{\pi^2}{M^2\epsilon^2}\psi^2 \right) dxdy, \qquad \psi \in H^{1,d}(\Omega_\epsilon). \tag{2.5}$$

We denote the corresponding operators as $\mathfrak{A}_{\epsilon,D}$ and $\mathfrak{A}_{\epsilon,DN}$, depending on the boundary condition at $x = -a$ and $x = b$. We suppress the subscripts D and DN when our argument applies to both operators. On their respective domains they act as

$$\mathfrak{A}_\epsilon \psi = \Delta\psi - \frac{\pi^2}{M^2\epsilon^2}\psi. \tag{2.6}$$

2.2. Limiting behavior of eigenvalues and eigenfunctions. It turns out that under the conditions (i), (ii) this behavior is determined by the operator on $L^2(\mathbb{R})$ given by

$$\mathbf{H} = -\frac{d^2}{dx^2} + q(x), \qquad q(x) = \begin{cases} 2\pi^2 M^{-3} c_+ x^m, \ x > 0, \\ 2\pi^2 M^{-3} c_- |x|^m, \ x < 0. \end{cases} \tag{2.7}$$

The spectrum of $\mathbf{H}$ is discrete and consists of simple eigenvalues, which we denote by μ_j. The corresponding eigenfunctions $X_j(x)$, normalized by the conditions $\|X_j\|_{L^2(\mathbb{R})} = 1$, $X_j(x) > 0$ for large $x > 0$, decay as $|x| \to \infty$ superexponentially fast. If $m = 2$ and $c_+ = c_- = c$, then $\mathbf{H}$ turns into the harmonic oscillator.

Theorem 2.1. *Let I be a finite segment and let $h(x)$ meet the conditions (i) and (ii). Then*

1) The eigenvalues $\lambda_j(\epsilon, D)$, $\lambda_j(\epsilon, DN)$ of the operators $\Delta_{\epsilon,D}$, $\Delta_{\epsilon,DN}$ have the same asymptotic behavior, namely

$$\lim_{\epsilon\to 0} \epsilon^{2\alpha} \left(\lambda_j(\epsilon) - \frac{\pi^2}{M^2\epsilon^2} \right) = \mu_j, \tag{2.8}$$

where

$$\alpha = 2(m+2)^{-1}. \tag{2.9}$$

2) For the normalized eigenfunctions $\Psi_j(\epsilon, D; x, y)$, $\Psi_j(\epsilon, DN; x, y)$ of the operators $\Delta_{\epsilon,D}$, $\Delta_{\epsilon,DN}$ we have, with an appropriate choice of sign:

$$\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} \left(\Psi_j(\epsilon; x, y) - \frac{\sqrt{2}}{\sqrt{\epsilon^{1+\alpha} h(x)}} X_j(x\epsilon^{-\alpha}) \right)^2 dxdy = 0.$$

For the D-case this is the result of Theorems 1.1 and 1.4 in [**3**]. For the DN-case the results are new.

Our proof of Theorem 2.1 is based upon the study of the operator family $\mathfrak{A}_\epsilon^{-1}$ as $\epsilon \to 0$. It consists of two steps. Firstly, we reduce the original problem to the one for an auxiliary ordinary differential operator $\mathbf{Q}_\epsilon$ acting on $L^2(I)$. Secondly, we show that the operators $(\epsilon^{2\alpha}\mathbf{Q}_\epsilon)^{-1}$ approach a family of operators on $L^2(\mathbb{R})$ that are unitarily equivalent to, and hence isospectral with, the operator $\mathbf{H}^{-1}$. In the next two subsections we describe these steps and formulate the corresponding results. Their proofs are given in Section 4; the general scheme is explained in Section 3. In Section 5 we show that under mild additional assumptions about the behavior of $h(x)$ as $|x| \to \infty$ this scheme applies also to the case $I = \mathbb{R}$. In the concluding Section 6 we prove that these conditions can be simplified even further if one is interested only in the behavior of the eigenvalues.

2.3. Reduction of dimension. In $L^2(\Omega_\epsilon)$ we take the subspace $\mathcal{L}_\epsilon$ that consists of the functions

$$\psi(x,y) = \psi_{\epsilon,\chi}(x,y) = \chi(x)\sqrt{\frac{2}{\epsilon h(x)}}\,\sin\frac{\pi y}{\epsilon h(x)}.$$

The mapping

$$\mathbf{\Pi}_\epsilon : \chi \mapsto \psi_{\epsilon,\chi} \tag{2.10}$$

is an isometric isomorphism of $L^2(I)$ onto $\mathcal{H}_\epsilon$, and we identify any operator $\mathbf{T}$ on $\mathcal{H}_\epsilon$ with the operator $\mathbf{\Pi}_\epsilon^{-1}\mathbf{T}\mathbf{\Pi}_\epsilon$ acting on $L^2(I)$. Let $\chi \in H^1(I)$. A direct computation shows that

$$\mathbf{a}_\epsilon[\psi_{\epsilon,\chi}] = \mathbf{q}_\epsilon[\chi] := \int_I \left(\chi'(x)^2 + W_\epsilon(x)\chi^2(x)\right)dx, \tag{2.11}$$

where

$$W_\epsilon(x) = \frac{\pi^2}{\epsilon^2}\left(\frac{1}{h^2(x)} - \frac{1}{M^2}\right) + \left(\frac{\pi^2}{3} + \frac{1}{4}\right)\frac{h'(x)^2}{h^2(x)}. \tag{2.12}$$

The quadratic form $\mathbf{q}_\epsilon[\chi]$, considered on $H^1(I)$, is positive definite and closed in $L^2(I)$. The same is true for its restriction to $H^{1,0}(I)$. The corresponding selfadjoint operator on $L^2(I)$ acts as

$$\mathbf{Q}_\epsilon\chi = -\chi'' + W_\epsilon(x)\chi, \tag{2.13}$$

with the Dirichlet or the Neumann condition at ∂I. When it is necessary to reflect it in the notation, we denote these operators by $\mathbf{Q}_{\epsilon,D}$ and $\mathbf{Q}_{\epsilon,N}$ respectively.

Below $\mathcal{L}^\epsilon$ stands for the orthogonal complement of $\mathcal{L}_\epsilon$ in $L^2(\Omega_\epsilon)$. Given a Hilbert space $\mathfrak{Z}$, we write $\mathbf{0}_\mathfrak{Z}$ for the zero operator on $\mathfrak{Z}$.

THEOREM 2.2. *Under the assumptions of Theorem* 2.1 *one has*

$$\left\|\mathfrak{A}_\epsilon^{-1} - \mathbf{Q}_\epsilon^{-1} \oplus \mathbf{0}_{\mathcal{L}^\epsilon}\right\| = O(\epsilon^{3\alpha}), \qquad \epsilon \to 0, \tag{2.14}$$

where $\mathfrak{A}_\epsilon$ is either of the operators $\mathfrak{A}_{\epsilon,D}$ and $\mathfrak{A}_{\epsilon,DN}$, and $\mathbf{Q}_\epsilon = \mathbf{Q}_{\epsilon,D}$ in the D-case and $\mathbf{Q}_\epsilon = \mathbf{Q}_{\epsilon,N}$ in the DN-case.

For the D-case this is Theorem 1.2 in [**3**].

2.4. From $\mathbf{Q}_\epsilon$ to $\mathbf{H}$. Along with the operator $\mathbf{H}$ defined in (2.7), let us consider on $L^2(\mathbb{R})$ the operator family

$$\mathbf{H}_\epsilon = -\frac{d^2}{dx^2} + \epsilon^{-2} q(x), \quad \epsilon > 0.$$

In particular $\mathbf{H}_1 = \mathbf{H}$. The substitution $x = t\epsilon^\alpha$ shows that for any $\epsilon > 0$ the operator $\epsilon^{2\alpha}\mathbf{H}_\epsilon$ is unitarily equivalent to $\mathbf{H}$, and hence, $\epsilon^{2\alpha}\mathbf{H}_\epsilon$ is an isospectral family of operators.

THEOREM 2.3. *Let $\mathbf{Q}_\epsilon$ be either of the operators $\mathbf{Q}_{\epsilon,D}$ and $\mathbf{Q}_{\epsilon,N}$. One has*

$$\lim_{\epsilon \to 0} \|(\epsilon^{2\alpha}\mathbf{Q}_\epsilon)^{-1} \oplus \mathbf{0}_{L^2(\mathbb{R}\setminus I)} - (\epsilon^{2\alpha}\mathbf{H}_\epsilon)^{-1}\| = 0.$$

For the D-case this is a reformulation of Theorem 1.3 in [**3**]; see (1.9) therein.

Note that Theorems 2.2 and 2.3 give a stronger result than Theorem 2.1. For instance, they imply that the convergence

$$\epsilon^{-2\alpha}\left(\lambda_j(\epsilon) - \frac{\pi^2}{M^2\epsilon^2}\right)^{-1} \to \mu_j^{-1},$$

cf. (2.8), is uniform in j.

The derivation of Theorem 2.1 from Theorems 2.2 and 2.3 was explained in [**3**], and we do not reproduce it here.

3. Proof of Theorem 2.2: General scheme

For the D-case the proof is given in [**3**], Section 3. For the DN-case the scheme remains the same, and we find it useful to present it in an abstract form.

Let $\mathfrak{H}_\epsilon$, $0 < \epsilon \le \epsilon_0$, be a family of separable Hilbert spaces. Keeping in mind our original problem, it is sufficient to consider them real. Let $\mathcal{H}_\epsilon \subset \mathfrak{H}_\epsilon$ be a family of their (closed) subspaces. We denote by $\mathcal{H}^\epsilon$ the orthogonal complement of $\mathcal{H}_\epsilon$ in $\mathfrak{H}_\epsilon$, so that

$$\mathfrak{H}_\epsilon = \mathcal{H}_\epsilon \oplus \mathcal{H}^\epsilon.$$

Below $\mathbf{P}_\epsilon$ and $\mathbf{P}^\epsilon$ stand for the orthogonal projections in $\mathfrak{H}_\epsilon$ onto the subspaces $\mathcal{H}_\epsilon$ and $\mathcal{H}^\epsilon$ respectively, and for an arbitrary element $\psi \in \mathfrak{H}_\epsilon$ we standardly write

$$\psi_\epsilon = \mathbf{P}_\epsilon \psi, \qquad \psi^\epsilon = \mathbf{P}^\epsilon \psi.$$

For each ϵ, let $\mathfrak{a}_\epsilon[\psi_1, \psi_2]$ be a symmetric bilinear form in $\mathfrak{H}_\epsilon$, defined for ψ_1, ψ_2 lying in a dense domain $\mathfrak{d}_\epsilon$. We suppose that the corresponding quadratic form $\mathfrak{a}[\psi] := \mathfrak{a}_\epsilon[\psi, \psi]$ is nonnegative and closed. We write $\mathfrak{A}_\epsilon$ for the selfadjoint operator on $\mathfrak{H}_\epsilon$, generated by $\mathfrak{a}_\epsilon$.

Suppose that

$$\psi \in \mathfrak{d}_\epsilon \implies \psi_\epsilon \in \mathfrak{d}_\epsilon.$$

Then also $\psi \in \mathfrak{d}_\epsilon \implies \psi^\epsilon \in \mathfrak{d}_\epsilon$ and the sets

$$\mathbf{d}_\epsilon := \{\psi_\epsilon : \psi \in \mathfrak{d}_\epsilon\}, \qquad \mathbf{d}^\epsilon := \{\psi^\epsilon : \psi \in \mathfrak{d}_\epsilon\}$$

are dense in $\mathcal{H}_\epsilon$ and in $\mathcal{H}^\epsilon$, respectively. Indeed, if, say, $\widetilde{\psi} \in \mathcal{H}_\epsilon$ and $(\psi_\epsilon, \widetilde{\psi}) = 0$ for all $\psi_\epsilon \in \mathbf{d}_\epsilon$, then also $(\psi, \widetilde{\psi}) = 0$ for all $\psi \in \mathfrak{d}_\epsilon$, and hence $\widetilde{\psi} = 0$.

Let us consider the families $\mathbf{a}_\epsilon = \mathfrak{a}_\epsilon \restriction \mathbf{d}_\epsilon$ and $\mathbf{a}^\epsilon = \mathfrak{a}_\epsilon \restriction \mathbf{d}^\epsilon$ of quadratic forms in $\mathcal{H}_\epsilon$ and in $\mathcal{H}^\epsilon$. Both $\mathbf{a}_\epsilon$ and $\mathbf{a}^\epsilon$ are closed. We denote by $\mathcal{A}_\epsilon$ and $\mathcal{A}^\epsilon$ the corresponding selfadjoint operators on $\mathcal{H}_\epsilon$ and on $\mathcal{H}^\epsilon$.

The quadratic form $\mathfrak{a}_\epsilon$ decomposes as

$$\mathfrak{a}_\epsilon[\psi] = \mathbf{a}_\epsilon[\psi_\epsilon] + \mathbf{a}^\epsilon[\psi^\epsilon] + 2\mathfrak{a}_\epsilon[\psi_\epsilon, \psi^\epsilon], \qquad \psi \in \mathfrak{d}_\epsilon. \tag{3.1}$$

Now, we make the following assumptions about the behavior of each term in (3.1).

$$\mathbf{a}_\epsilon[\psi_\epsilon] \geq c(\epsilon)\|\psi_\epsilon\|^2, \qquad \forall \psi_\epsilon \in \mathbf{d}_\epsilon, \qquad c(\epsilon) \geq c_0 > 0; \tag{3.2}$$

$$\mathbf{a}^\epsilon[\psi^\epsilon] \geq p(\epsilon)\|\psi^\epsilon\|^2, \qquad \forall \psi^\epsilon \in \mathbf{d}^\epsilon; \tag{3.3}$$

$$p(\epsilon) \to \infty, \qquad c(\epsilon) = O(p(\epsilon)); \tag{3.4}$$

$$|\mathfrak{a}_\epsilon[\psi_\epsilon, \psi^\epsilon]|^2 \leq q^2(\epsilon)\mathbf{a}_\epsilon[\psi_\epsilon]\mathfrak{a}_\epsilon[\psi^\epsilon], \qquad \forall \psi \in \mathfrak{d}_\epsilon, \qquad q(\epsilon) \to 0. \tag{3.5}$$

PROPOSITION 3.1. *Let the conditions* (3.2)–(3.5) *be satisfied. Then for ϵ small enough the operator $\mathfrak{A}_\epsilon$ is positive definite, and*

$$\|\mathfrak{A}_\epsilon^{-1} - \mathcal{A}_\epsilon^{-1} \oplus \mathbf{0}_{\mathcal{H}^\epsilon}\| \leq p(\epsilon)^{-1} + Cq(\epsilon)c(\epsilon)^{-1}. \tag{3.6}$$

PROOF. Along with $\mathfrak{a}_\epsilon[\psi]$, let us consider its diagonal part, i.e. the quadratic form

$$\widehat{\mathfrak{a}}_\epsilon[\psi] = \mathbf{a}_\epsilon[\psi_\epsilon] + \mathbf{a}^\epsilon[\psi^\epsilon], \qquad \psi \in \mathfrak{d}_\epsilon.$$

By (3.2) and (3.3), $\widehat{\mathfrak{a}}_\epsilon$ is positive definite. It is also closed, since both its components are closed. Let $\widehat{\mathfrak{A}}_\epsilon$ be the corresponding selfadjoint operator on $\mathfrak{H}_\epsilon$. By (3.4), we have for small ϵ:

$$\|\widehat{\mathfrak{A}}_\epsilon^{-1}\| \leq Cc(\epsilon)^{-1},$$

with some constant $C > 0$. Besides, (3.5) implies

$$|\mathfrak{a}_\epsilon[\psi_\epsilon, \psi^\epsilon]| \leq q(\epsilon)\widehat{\mathfrak{a}}_\epsilon[\psi]$$

and hence,

$$|\mathfrak{a}_\epsilon[\psi] - \widehat{\mathfrak{a}}_\epsilon[\psi]| = 2|\mathfrak{a}_\epsilon[\psi_\epsilon, \psi^\epsilon]| \leq 2q(\epsilon)\widehat{\mathfrak{a}}_\epsilon[\psi].$$

For ϵ small enough, so that $q(\epsilon) \leq 1/4$, this implies

$$\tfrac{1}{2}\,\widehat{\mathfrak{a}}_\epsilon[\psi] \leq \mathfrak{a}_\epsilon[\psi] \leq \tfrac{3}{2}\,\widehat{\mathfrak{a}}_\epsilon[\psi], \qquad \forall \psi \in \mathfrak{d}_\epsilon.$$

Hence, for such an ϵ the operator $\mathfrak{A}_\epsilon$ is positive definite (rather than only nonnegative, as was originally assumed), and

$$\|\mathfrak{A}_\epsilon^{-1}\| \leq 2Cc(\epsilon)^{-1}.$$

Note also that (3.3) is equivalent to

$$\|(\mathcal{A}^\epsilon)^{-1}\| \leq p(\epsilon)^{-1}. \tag{3.7}$$

Further, for any $\psi_1, \psi_2 \in \mathfrak{d}_\epsilon$ and ϵ small enough, one has

$$\begin{aligned}
\Big|(\mathfrak{A}_\epsilon^{1/2}\psi_1, \mathfrak{A}_\epsilon^{1/2}\psi_2) - (\widehat{\mathfrak{A}}_\epsilon^{1/2}\psi_1, \widehat{\mathfrak{A}}_\epsilon^{1/2}\psi_2)\Big| &= |\mathfrak{a}_\epsilon[\psi_1, \psi_2] - \widehat{\mathfrak{a}}_\epsilon[\psi_1, \psi_2]| \\
= |\mathfrak{a}_\epsilon[\psi_{1,\epsilon}, \psi_2^\epsilon] + \mathfrak{a}_\epsilon[\psi_1^\epsilon, \psi_{2,\epsilon}]| &\leq q(\epsilon)\left((\mathbf{a}_\epsilon[\psi_{1,\epsilon}]\mathfrak{a}_\epsilon[\psi_2^\epsilon])^{1/2} + (\mathbf{a}_\epsilon[\psi_{2,\epsilon}]\mathfrak{a}_\epsilon[\psi_1^\epsilon])^{1/2}\right) \\
&\leq 2q(\epsilon)(\widehat{\mathfrak{a}}_\epsilon[\psi_1]\widehat{\mathfrak{a}}_\epsilon[\psi_2])^{1/2} \leq 2\sqrt{2}\,q(\epsilon)(\widehat{\mathfrak{a}}_\epsilon[\psi_1]\mathfrak{a}_\epsilon[\psi_2])^{1/2}.
\end{aligned}$$

In the last formula we take $\psi_1 = \widehat{\mathfrak{A}}_\epsilon^{-1}f,\ \psi_2 = \mathfrak{A}_\epsilon^{-1}g$; here $f, g \in \mathfrak{H}_\epsilon$ are arbitrary elements. Then

$$\begin{aligned}
&|(\widehat{\mathfrak{A}}_\epsilon^{-1}f, g) - (\mathfrak{A}_\epsilon^{-1}f, g)| \\
&\leq 2\sqrt{2}\,q(\epsilon)\big((\mathfrak{A}_\epsilon^{-1}g, g)(\widehat{\mathfrak{A}}_\epsilon^{-1}f, f)\big)^{1/2} \leq 4Cq(\epsilon)c(\epsilon)^{-1}\|f\|\|g\|,
\end{aligned}$$

and therefore

$$\|\mathfrak{A}_\epsilon^{-1} - \widehat{\mathfrak{A}}_\epsilon^{-1}\| \le 4Cq(\epsilon)c(\epsilon)^{-1}. \tag{3.8}$$

Since $\widehat{\mathfrak{A}}_\epsilon^{-1} = \mathcal{A}_\epsilon^{-1} \oplus (\mathcal{A}^\epsilon)^{-1}$, we conclude that

$$\|\widehat{\mathfrak{A}}_\epsilon^{-1} - \mathcal{A}_\epsilon^{-1} \oplus \mathbf{0}_{\mathcal{H}^\epsilon}\| = \|(\mathcal{A}^\epsilon)^{-1}\|.$$

Together with (3.7) and (3.8), this leads to (3.6). □

4. Proof of Theorems 2.2 and 2.3 (the DN-case)

4.1. Preliminaries. To prove Theorem 2.2, we use Proposition 3.1 with $\mathfrak{H}_\epsilon = L^2(\Omega_\epsilon)$ and $\mathcal{H}_\epsilon = \mathcal{L}_\epsilon$. As in Section 2.3, we identify operators $\mathbf{T}$ acting in $\mathcal{L}_\epsilon$ with their images $\mathbf{\Pi}_\epsilon^{-1}\mathbf{T}\mathbf{\Pi}_\epsilon$ acting in $L^2(I)$. Here $\mathbf{\Pi}_\epsilon$ is the isometry given by (2.10). The basic bilinear form is

$$\mathfrak{a}_\epsilon[\psi_1, \psi_2] = \int_{\Omega_\epsilon} \left(\frac{\partial \psi_1}{\partial x}\frac{\partial \psi_2}{\partial x} + \frac{\partial \psi_1}{\partial y}\frac{\partial \psi_2}{\partial y} - \frac{\pi^2}{M^2\epsilon^2}\psi_1\psi_2 \right) dxdy \tag{4.1}$$

defined for $\psi_1, \psi_2 \in \mathfrak{d}_\epsilon := H^{1,d}(\Omega_\epsilon)$. The corresponding quadratic form is given by (2.5). For each $\psi \in \mathbf{d}_\epsilon = H^{1,d}(\Omega_\epsilon) \cap \mathcal{L}_\epsilon$ there exists one and only one $\chi \in H^1(I)$ such that $\psi_\epsilon = \psi_{\epsilon,\chi}$, and then $\mathbf{a}_\epsilon[\psi_\epsilon] = \mathbf{q}_\epsilon[\chi]$. It is immediate that the operator $\mathcal{A}_\epsilon$ is nothing but $\mathbf{Q}_{\epsilon,DN}$, and hence, the proof of Theorem 2.2 for the DN-case reduces to proving the inequalities (3.2)–(3.5) with appropriate constants $c(\epsilon), p(\epsilon)$ and $q(\epsilon)$.

4.2. Proof of (3.2)**.** The inequality (3.2) is the only inequality, the proof of which requires a new argument compared with [**3**]. There are several ways to prove (3.2). We choose a way that is based on a remarkable result due to Birman; see [**1**], [**2**]. This result belongs to the general theory of selfadjoint extensions of symmetric operators.

The inequality (3.2) can be rewritten as

$$\mathbf{q}_\epsilon[\chi] \ge c\epsilon^{-2\alpha}\|\chi\|^2, \qquad \chi \in H^1(I),$$

or equivalently,

$$\|\mathbf{Q}_{\epsilon,N}^{-1}\| \le c^{-1}\epsilon^{2\alpha}. \tag{4.2}$$

A similar inequality for $\mathbf{Q}_{\epsilon,D}^{-1}$ was proved in [**3**], Lemma 2.1. Therefore, it is sufficient to estimate the norm of the operator $\mathbf{Q}_{\epsilon,N}^{-1} - \mathbf{Q}_{\epsilon,D}^{-1}$. For technical reasons, it is more convenient to deal with the operator

$$\mathbf{S}_\epsilon = (\mathbf{Q}_{\epsilon,N} + 1)^{-1} - (\mathbf{Q}_{\epsilon,D} + 1)^{-1}.$$

Its norm can be estimated by using lemma 3.1 in [**2**]. When applied to the operators in question, it yields

$$\|\mathbf{S}_\epsilon\| = \max\left\{\mathcal{R}(u) : \ u \in H^1(I),\ -u'' + (1 + W_\epsilon(x))u = 0\right\}, \tag{4.3}$$

where

$$\mathcal{R}(u) = \frac{\int_I |u|^2 dx}{\int_I (|u'|^2 + (1 + W_\epsilon(x))|u|^2)\, dx}.$$

The weak form of the differential equation from (4.3) is

$$\int_I (u'\phi' + (1 + W_\epsilon(x))u\phi)\, dx = 0, \qquad \forall \phi \in H^{1,0}(I). \tag{4.4}$$

Take $\phi(x) = u(x)\zeta_{ab}(x)$; the function $\zeta_{ab}(x)$ is described below. First, we fix a function $\zeta \in C^\infty(0,2)$ such that

$$\zeta(t) = 1,\ t \le 1;\ \zeta(t) = (2-t)^2,\ t > 3/2;\ 0 \le \zeta(t) \le 1 \text{ everywhere}.$$

Denote $K = \max \zeta'(t)^2/\zeta(t)$, and set

$$\zeta_{ab}(x) = \begin{cases} b^2\zeta(2x/b), & x \in [0,b); \\ a^2\zeta(2|x|/a), & x \in (-a,0]. \end{cases}$$

We have $\zeta_{ab} \in C^\infty(I)$, $\zeta(-a) = \zeta(b) = 0$, and $\zeta'_{ab}(x)^2 \le 4K\zeta_{ab}(x)$. The function $\phi = u\zeta_{ab}$ lies in $H^{1,0}(I)$, and we conclude from (4.4) that

$$2\int_I \left({u'}^2 + (1+W_\epsilon(x))u^2\right)\zeta_{ab}dx = -2\int_I u'u\zeta'_{ab}dx$$
$$\le \int_I {u'}^2\zeta_{ab}dx + \int_I u^2\frac{(\zeta'_{ab})^2}{\zeta_{ab}}dx \le \int_I {u'}^2\zeta_{ab}dx + 4K\int_{\widetilde I} u^2dx,$$

where $\widetilde I = (-a,-a/2)\cup(b/2,b)$. From here we derive that

$$\int_I \left({u'}^2 + 2(1+W_\epsilon(x))u^2\right)\zeta_{ab}dx \le 4K\int_{\widetilde I} u^2dx.$$

Hence,

$$\int_{a/2}^{b/2} u^2dx \le 2K\int_{\widetilde I} u^2dx.$$

The conditions on $h(x)$ imply the inequality

$$W_\epsilon(x) \ge \sigma\epsilon^{-2}|x|^m, \qquad \forall\epsilon > 0,\ x \in I,$$

with some $\sigma > 0$; see (2.2) in [**3**]. This yields

$$\mathcal{R}(u) \le (1+2K)\frac{\int_{\widetilde I} u^2dx}{\int_{\widetilde I} W_\epsilon(x)u^2dx} \le \frac{1+2K}{\sigma(\min(a,b)/2)^m}\epsilon^2.$$

So, $\|\mathbf{S}_\epsilon\| = O(\epsilon^2)$. From the Hilbert resolvent formula we conclude that also

$$\|\mathbf{Q}_{\epsilon,N}^{-1} - \mathbf{Q}_{\epsilon,D}^{-1}\| = O(\epsilon^2). \tag{4.5}$$

By (2.9), $\alpha \le 2/3$ and hence, $\|\mathbf{Q}_{\epsilon,D}^{-1}\| = O(\epsilon^{2\alpha})$ yields the same estimate for $\|\mathbf{Q}_{\epsilon,N}^{-1}\|$. This completes the proof of (4.2) and hence, that of (3.2), with $c(\epsilon) = C\epsilon^{-2\alpha}$.

4.3. Proof of (3.3)**.** This inequality is the easiest to prove. The inclusion $\psi \in \mathcal{H}^\epsilon$ means that

$$\int_0^{\epsilon h(x)} \psi(x,y)\sin\frac{\pi y}{\epsilon h(x)}dy = 0, \qquad \text{a.a. } x \in I. \tag{4.6}$$

In other words, the function $\psi(x,\cdot)$ is orthogonal to the first eigenfunction of the operator $-u''_{yy}$ on the interval $(0,\epsilon h(x))$, with the Dirichlet boundary conditions at its ends. Thus, every function $\psi \in \mathfrak{d}_\epsilon \cap \mathcal{H}^\epsilon$ satisfies an estimate similar to (2.3); its right-hand side contains an additional factor 4. Therefore

$$\|\psi\|^2 \le \frac{M^2\epsilon^2}{3\pi^2}\mathfrak{a}_\epsilon[\psi], \qquad \forall\psi \in \mathbf{d}^\epsilon,$$

which means that (3.3) is satisfied with

$$p(\epsilon) = 3\pi^2M^{-2}\epsilon^{-2}. \tag{4.7}$$

In particular, (4.7) implies (3.4). If $m > 1$ in (2.1), then the symbol O in (3.4) can be replaced by o.

4.4. End of the proof of Theorem 2.2. The proof of (3.5) repeats the proof of the similar inequality in [**3**]. Nevertheless, we outline the argument.

For $\psi \in \mathbf{d}^\epsilon$ we can integrate (4.6) by parts in y and differentiate it in x. This results in the equalities that hold for a.a. $x \in I$:

$$\int_0^{\epsilon h(x)} \psi'_y(x,y) \cos \frac{\pi y}{\epsilon h(x)} dy = 0;$$

$$\int_0^{\epsilon h(x)} \psi'_x(x,y) \sin \frac{\pi y}{\epsilon h(x)} dy = \frac{\pi}{\epsilon} \frac{h'(x)}{h^2(x)} \int_0^{\epsilon h(x)} y\psi(x,y) \cos \frac{\pi y}{\epsilon h(x)} dy.$$

Let $\psi \in \mathfrak{d}_\epsilon$ and $\psi_\epsilon = \psi_{\epsilon,\chi}$ with some $\chi \in H^1(I)$. The off-diagonal part of $\mathfrak{a}_\epsilon[\psi]$ reduces to the form

$$\mathfrak{a}_\epsilon[\psi_\epsilon, \psi^\epsilon] = \int_{\Omega_\epsilon} (\psi_{\epsilon,\chi})'_x (\psi^\epsilon)'_x dxdy \tag{4.8}$$

$$= \frac{\sqrt{2}\pi}{\epsilon^{3/2}} \int_{\Omega_\epsilon} \widetilde{h}(x) \cos \frac{\pi y}{\epsilon h(x)} \left(\phi' \psi - \phi \psi'_x\right) y dxdy,$$

where

$$\widetilde{h}(x) = \frac{h'(x)}{h^2(x)}, \qquad \phi(x) = \frac{\chi(x)}{h^{1/2}(x)}.$$

Now the estimate (3.5) with $q(\epsilon) = C\epsilon^\alpha$ follows by the Cauchy-Schwarz inequality.

Applying the estimate (3.6) to the operator $\mathfrak{A}_{\epsilon,DN}$, we conclude that

$$\left\|\mathfrak{A}_\epsilon^{-1} - \mathbf{Q}_\epsilon^{-1} \oplus \mathbf{0}_{\mathcal{L}^\epsilon}\right\| \le C(\epsilon^2 + \epsilon^{3\alpha});$$

together with $\alpha \le 2/3$, the last estimate implies (2.14). This completes the proof of Theorem 2.2.

4.5. Proof of Theorem 2.3. For the D-case this is an equivalent reformulation of Theorem 1.3 in [**3**]; see (1.10) there. In the DN-case, we apply the inequality (4.5), which yields

$$\|(\epsilon^{2\alpha}\mathbf{Q}_{\epsilon,N})^{-1} \oplus \mathbf{0}_{L^2(\mathbb{R}\setminus I)} - (\epsilon^{2\alpha}\mathbf{H}_\epsilon)^{-1}\|$$
$$\le \|(\epsilon^{2\alpha}\mathbf{Q}_{\epsilon,D})^{-1} \oplus \mathbf{0}_{L^2(\mathbb{R}\setminus I)} - (\epsilon^{2\alpha}\mathbf{H}_\epsilon)^{-1}\| + \epsilon^{-2\alpha}\|\mathbf{Q}_{\epsilon,N}^{-1} - \mathbf{Q}_{\epsilon,D}^{-1}\|.$$

Both terms on the right tend to zero as $\epsilon \to 0$: the first by Theorem 1.3 in [**3**] and the second by (4.5), again keeping in mind that $\alpha < 2/3$. This completes the proof of Theorem 2.3.

5. The case $I = \mathbb{R}$: Behavior of the resolvent

In this and the next section we study the case $I = \mathbb{R}$. The spectrum of the Dirichlet Laplacian Δ_ϵ in Ω_ϵ is now not necessarily discrete, the structure of its essential component $\sigma_{ess}(\Delta_\epsilon)$ depends on the behavior of $h(x)$ at infinity. In this section we study the case when $h(x)$ satisfies conditions (i) and (ii), and also the following additional conditions:

(iii) $\limsup\limits_{|x|\to\infty} h(x) < M$,

(iii') $h'/h \in L^\infty(\mathbb{R})$.

Our goal is to show that under these conditions, analogues of Theorems 2.2 and 2.3 hold. A result similar to Theorem 2.1 then follows automatically, though its formulation becomes a bit more complicated because the operator Δ_ϵ can have a nonempty essential spectrum.

The situation simplifies if one is interested only in the behavior of the eigenvalues. In the next section 6 we will give a short proof of an analogue of statement 1) in Theorem 2.1. For that purpose, the condition (iii') turns out to be not necessary.

5.1. Behavior of the resolvent: formulations and auxiliary results. Let $I = \mathbb{R}$. The quadratic form (2.11) is positive definite and closed on the natural domain

$$\mathbf{d}_\epsilon = \{\chi \in H^1(\mathbb{R}) : \mathbf{q}_\epsilon[\chi] < \infty\}.$$

As before, the corresponding selfadjoint operator $\mathbf{Q}_\epsilon$ is formally given by (2.13). Note also that instead of two operators $\mathfrak{A}_{\epsilon,D},\ \mathfrak{A}_{\epsilon,DN}$ we have only one operator $\mathfrak{A}_\epsilon$.

THEOREM 5.1. *Let $I = \mathbb{R}$ and let $h(x)$ satisfy the conditions (i)–(iii$'$). Then the equality* (2.14) *holds.*

For the analogue of Theorem 2.3 we do not need condition (iii').

THEOREM 5.2. *Let $I = \mathbb{R}$ and let $h(x)$ satisfy the conditions (i)–(iii). Then*

$$\lim_{\epsilon\to 0} \|(\epsilon^{2\alpha}\mathbf{Q}_\epsilon)^{-1} - (\epsilon^{2\alpha}\mathbf{H}_\epsilon)^{-1}\| = 0. \tag{5.1}$$

As in Section 4, in the proofs of these theorems we rely upon Proposition 3.1, taking $\mathfrak{H}_\epsilon = L^2(\Omega_\epsilon)$ and $\mathcal{H}_\epsilon = \mathcal{L}_\epsilon$. The basic bilinear form is again given by (4.1). We have $\mathcal{A}_\epsilon = \mathbf{Q}_\epsilon$. The latter operator acts on $L^2(\mathbb{R})$; it is generated by the quadratic form (2.11) defined on its natural domain. The most important thing is to prove (5.1). Indeed, the conditions (i)–(iii) may lead to a function $W_\epsilon(x)$ that is bounded; then Theorem 2.16 from the book [**8**], which was the main ingredient of our proof of Theorem 1.3 in [**3**], does not apply. We need an appropriate substitute. First, we prove the following lemma, which can be considered as a partial generalization of Theorem 2.16 in [**8**]. By $\mathcal{C}$ we denote the ideal of all compact operators in the algebra of all bounded operators.

PROPOSITION 5.3. *Let $\mathbf{T} \geq 0$ and $\mathbf{T}_\epsilon \geq 0,\ 0 < \epsilon \leq \epsilon_0$, be bounded selfadjoint operators in a separable Hilbert space $\mathfrak{H}$, and let $\mathbf{T}_\epsilon \to \mathbf{T}$ strongly as $\epsilon \to 0$. Suppose also that there exists a bounded selfadjoint operator $\mathbf{T}_0$ such that $\mathbf{T}_\epsilon \leq \mathbf{T}_0$ for all $\epsilon \leq \epsilon_0$, and*

$$\operatorname{dist}(\mathbf{T}_0, \mathcal{C}) = m, \qquad m \geq 0.$$

Then

$$\limsup_{\epsilon\to 0} \|\mathbf{T}_\epsilon - \mathbf{T}\| \leq m.$$

PROOF. Fix $\eta > 0$ and find an operator $\mathbf{S} = \mathbf{S}^* \in \mathcal{C}$ that satisfies $\|\mathbf{T}_0 - \mathbf{S}\| < m + \eta$. Let $\mathfrak{F}$ be a finite-dimensional subspace in $\mathfrak{H}$, such that $\|\mathbf{S}g\| \leq \eta\|g\|$ for any $g \perp \mathfrak{F}$. Then

$$\|\mathbf{T}_0 g\| \leq \|(\mathbf{T}_0 - \mathbf{S})g\| + \|\mathbf{S}g\| \leq (m + 2\eta)\|g\|, \qquad \forall g \perp \mathfrak{F}.$$

Now, let $h \in \mathfrak{H}$ be an arbitrary element, and let $h = f + g$ where $f \in \mathfrak{F}$ and $g \perp \mathfrak{F}$. Then

$$((\mathbf{T}_\epsilon - \mathbf{T})h, h) = ((\mathbf{T}_\epsilon - \mathbf{T})f, f) + ((\mathbf{T}_\epsilon - \mathbf{T})g, g) + 2\operatorname{Re}\left((\mathbf{T}_\epsilon - \mathbf{T})f, g\right).$$

Since the dimension of $\mathfrak{F}$ is finite, strong convergence $\mathbf{T}_\epsilon \to \mathbf{T}$ implies

$$\lim_{\epsilon\to 0} \sup_{f\in\mathfrak{F}} \|(\mathbf{T}_\epsilon - \mathbf{T})f\|/\|f\| = 0.$$

Therefore,

$$|\,((\mathbf{T}_\epsilon - \mathbf{T})h, h)\,| \le |\,((\mathbf{T}_\epsilon - \mathbf{T})g, g)\,| + \eta\|f\|^2 + 2\eta\|f\|\|g\|$$

if ϵ is small enough. Further,

$$-(\mathbf{T}_0 g, g) \le ((\mathbf{T}_\epsilon - \mathbf{T})g, g) \le (\mathbf{T}_0 g, g),$$

since both $\mathbf{T}$ and $\mathbf{T}_\epsilon$ are nonnegative operators. Therefore,

$$|\,((\mathbf{T}_\epsilon - \mathbf{T})g, g)\,| \le (\mathbf{T}_0 g, g) \le (m + 2\eta)\|g\|^2.$$

Finally, this yields

$$|\,((\mathbf{T}_\epsilon - \mathbf{T})h, h)\,| \le (m + 2\eta)\|g\|^2 + 2\eta\|f\|^2 + \eta\|g\|^2 \le (m + 3\eta)\|h\|^2.$$

Since $\eta > 0$ is arbitrary, the statement of the proposition follows. $\square$

Below, the symbol $\mathbf{Z}_V$ denotes the Schrödinger operator on $L^2(\mathbb{R})$ with the nonnegative potential $V(x)$. We define the operator $\mathbf{Z}_V$ via its quadratic form. The next statement is a substitute for proposition 4.1 in [**3**].

Proposition 5.4. *Let $V(x)$ and $V_\epsilon(x)$, $0 < \epsilon \le \epsilon_0$, be nonnegative measurable functions on $\mathbb{R}$, such that $V(x) \to \infty$ as $|x| \to \infty$ and*

$$V_\epsilon(x) \to V(x) \text{ as } \epsilon \to 0, \qquad \text{uniformly on compact sets.} \tag{5.2}$$

Suppose also that

$$V_\epsilon(x) \ge V_\epsilon^\circ(x), \qquad \forall x \in \mathbb{R},\ 0 < \epsilon \le \epsilon_0,$$

where $V_\epsilon^\circ(x)$ is another family of measurable functions on $\mathbb{R}$, which is monotone in ϵ:

$$\epsilon_1 > \epsilon_2 \implies V_{\epsilon_1}^\circ(x) \le V_{\epsilon_2}^\circ(x), \qquad \forall x \in \mathbb{R} \tag{5.3}$$

and

$$c(\epsilon) := \liminf_{|x|\to\infty} V_\epsilon^\circ(x) \to \infty \qquad \text{as } \epsilon \to 0. \tag{5.4}$$

Then

$$\|\mathbf{Z}_{V_\epsilon}^{-1} - \mathbf{Z}_V^{-1}\| \to 0, \qquad \epsilon \to 0. \tag{5.5}$$

Proof. Denote

$$\mathbf{T} = \mathbf{Z}_V^{-1}, \qquad \mathbf{T}_\epsilon = \mathbf{Z}_{V_\epsilon}^{-1}.$$

Also let $\mathbf{T}_\epsilon^\circ$ be the inverse to the Schrödinger operator with the potential $V_\epsilon^\circ(x)$. The assumption (5.2) implies that $\mathbf{Z}_{V_\epsilon} u \to \mathbf{Z}_V u$ for any $u \in C_0^\infty(\mathbb{R})$, and Theorem 8.1.5 in [**5**] guarantees the strong convergence $\mathbf{T}_\epsilon \to \mathbf{T}$.

Fix $\epsilon^* \in (0, \epsilon_0)$; then for $\epsilon < \epsilon^*$ the conditions of Proposition 5.3 are satisfied with $\mathbf{T}_0 = \mathbf{T}_{\epsilon^*}^\circ$. From this proposition we conclude that

$$\limsup_{\epsilon\to 0} \|\mathbf{T}_\epsilon - \mathbf{T}\| \le 1/c(\epsilon^*).$$

Taking $\epsilon^* \to 0$, we arrive at (5.5). $\square$

5.2. Proof of Theorem 5.2. Introduce the potential

$$V_\epsilon(t) = \epsilon^{2\alpha} W_\epsilon(t\epsilon^\alpha),$$

where $W_\epsilon(x)$ is the function defined in (2.12). The assumption (2.1) implies that

$$V_\epsilon(t) = q(t) + \pi^2 \rho_1(t\epsilon^\alpha) t^{m+1} \epsilon^\alpha + \epsilon^{2\alpha} v(t\epsilon^\alpha),$$

where $\rho_1(x)$ is some function which is bounded on any finite interval $(-a, a)$; cf. proof of Theorem 1.3 in [**3**]. Hence, $V_\epsilon(t) \to q(t)$ uniformly on compact subsets in $\mathbb{R}$.

It follows from the assumptions (i) and (iii) that for $|x| \geq 1$ the function $\pi^2(h^{-2}(x) - M^{-2})$ is bounded below:

$$\pi^2(h^{-2}(x) - M^{-2}) \geq c_0, \qquad |x| \geq 1,$$

with some $c_0 > 0$. On $[-1, 1]$ the inequality $W_\epsilon(x) \geq \sigma\epsilon^{-2}|x|^m$ with some $\sigma > 0$ is fulfilled, which leads to the estimate

$$W_\epsilon(x) \geq \sigma_1 \epsilon^{-2} \min(|x|^m, 1), \qquad \forall x \in \mathbb{R},\ \epsilon > 0.$$

Hence,

$$V_\epsilon(t) \geq \sigma_1 \min(\epsilon^{-2+2\alpha+m\alpha}|t|^m, \epsilon^{-2+2\alpha}) = \sigma_1 \min(|t|^m, \epsilon^{-2+2\alpha}).$$

The conditions (5.3) and (5.4) are satisfied if we take

$$V_\epsilon^\circ(t) = \sigma_1 \min(|t|^m, \epsilon^{-2+2\alpha}).$$

So, Theorem 5.4 applies and yields

$$\|\widehat{\mathbf{Q}}_\epsilon^{-1} - \mathbf{H}^{-1}\| \to 0,$$

where $\widehat{\mathbf{Q}}_\epsilon$ is the Schrödinger operator with the potential $V_\epsilon(t)$. The substitution $t = x\epsilon^{-\alpha}$ leads to (5.1).

5.3. Theorem 5.1: Outline of proof. The argument necessary for proving Theorem 5.1 is quite similar to the one in [**3**], and also in Section 4 of the present paper; we only outline it, concentrating on the few new moments.

First, note that

$$\|(\epsilon^{2\alpha}\mathbf{H}_\epsilon)^{-1}\| = \|\mathbf{H}^{-1}\| = const,$$

and hence (5.1) yields

$$\|\mathbf{Q}_\epsilon^{-1}\| \leq C\epsilon^{2\alpha}.$$

Equivalently, this means that (3.2) is satisfied with $c(\epsilon) = C^{-1}\epsilon^{-2\alpha}$. The condition (3.3) is satisfied with the same $p(\epsilon)$ as in (4.7); its proof does not change. Since $\alpha \leq 2/3$, (3.4) also holds. It only remains to prove (3.5). To this end, we repeat the reasoning in Section 4.4. The only difference is that now for estimating the integral in the right-hand side of (4.8) we need the condition (iii').

6. The case $I = \mathbb{R}$: Behavior of eigenvalues, a simple proof

An analogue of Theorem 2.1 follows from Theorems 5.1 and 5.2 in the same way as Theorem 2.1 itself follows from Theorems 2.2 and 2.3. However, there is a much simpler and independent way to prove an analogue of statement 1) in Theorem 2.1. It uses the Dirichlet–Neumann bracketing, which is possible, since Theorem 2.1 is now at our disposal for both the D and the DN cases. For the proof, the condition (iii') is not needed.

In Theorem 6.1 below, $\nu(\epsilon)$ stands for the bottom of $\sigma_{ess}(\Delta_\epsilon)$, and we take $\nu(\epsilon) = \infty$ if Δ_ϵ has no essential spectrum. We denote by $n(\epsilon)$, $n(\epsilon) \le \infty$, the number of eigenvalues $\lambda_j(\epsilon) < \nu(\epsilon)$.

THEOREM 6.1. *Let $I = \mathbb{R}$. If $h(x)$ satisfies the conditions (i), (ii) and (iii), then $\nu(\epsilon) \to \infty$ as $\epsilon \to 0$. Moreover, for small values of ϵ the spectrum of Δ_ϵ below $\nu(\epsilon)$ is nonempty, $n(\epsilon) \to \infty$ as $\epsilon \to 0$, and for each $j \in \mathbb{N}$ the equality* (2.8) *holds; μ_j are eigenvalues of the operator* (2.7).

PROOF. Take any segment $\widehat{I} = [-a, a]$. It follows from the conditions (i) and (iii) that

$$M_a := \sup_{x \notin \widehat{I}} h(x) < M.$$

Denote

$$\Omega_{a,\epsilon} = \{(x, y) \in \Omega_\epsilon : x \in \widehat{I}\}, \qquad \Omega'_{a,\epsilon} = \{(x, y) \in \Omega_\epsilon : x \notin \widehat{I}\}.$$

Let $\Delta_{a,\epsilon,D}$, $\Delta'_{a,\epsilon,D}$ be the Dirichlet Laplacians in $\Omega_{a,\epsilon}$, $\Omega'_{a,\epsilon}$ respectively; $\Delta_{a,\epsilon,DN}$ and $\Delta'_{a,\epsilon,DN}$ have a similar meaning. Then

$$\Delta_{a,\epsilon,DN} + \Delta'_{a,\epsilon,DN} < \Delta'_\epsilon < \Delta_{a,\epsilon,D} + \Delta'_{a,\epsilon,D}.$$

For $\Omega'_{a,\epsilon}$ an inequality similar to (2.4), with M_a in place of M, is satisfied; it implies that the spectra of both operators $\Delta'_{a,\epsilon,D}$ and $\Delta'_{a,\epsilon,DN}$ lie above the number $C(\epsilon) := \frac{\pi^2}{M_a^2 \epsilon^2}$, which is greater than $\frac{\pi^2}{M^2\epsilon^2}$. It follows that $\nu(\epsilon) \ge C(\epsilon)$, and the eigenvalues $\lambda_j(\epsilon) = \lambda_j(\Delta_\epsilon)$ lying below $C(\epsilon)$ satisfy the two-sided inequality

$$\lambda_j(\Delta_{a,\epsilon,DN}) \le \lambda_j(\epsilon) \le \lambda_j(\Delta_{a,\epsilon,D}).$$

The asymptotics (2.8) for the eigenvalues $\lambda_j(\Delta_{a,\epsilon,DN})$, $\lambda_j(\Delta_{a,\epsilon,D})$ implies that the number $\nu(\epsilon)$ grows indefinitely as $\epsilon \to 0$, and that the eigenvalues $\lambda_j(\epsilon)$ have the same asymptotics given by (2.8). $\square$

Another problem of the same nature was analyzed in Section 6.1 of [**3**]. It concerns the case when the segment I is finite but the function $h(x)$ is allowed to vanish at the ends of I.

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