The tug-of-war without noise and the infinity Laplacian in a wedge

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Frontier Probability Days

(joint work with Dante DeBlassie)

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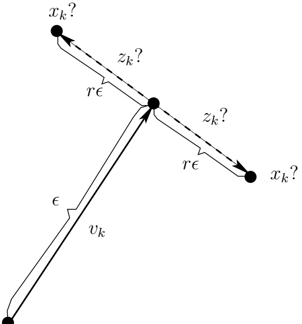

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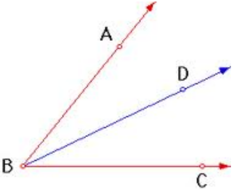

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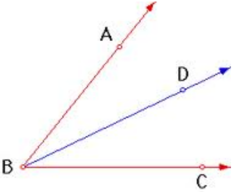

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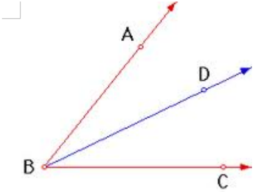

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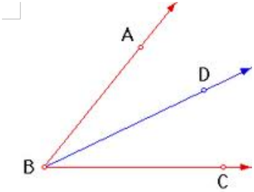

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- If $\eta > \eta_p$, then for each strategy of player II, there are arbitrarily small stepsizes, along with corresponding $x_0 \in W_\eta$ and strategy for player I such that

$$E_{x_0}[\tau] = \infty.$$

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The upper bound is weak, while the lower bound is what we believe is the correct behavior. The fact that $p < \infty$ gives some smoothness to solutions.

Tug of War without noise a la (Peres-Schramm-Sheffield-Wilson)

$$\Delta_\infty u = |\nabla u|^{-2} \left[\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right].$$

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The game position is initially at a point $x_0 \in D$ and at each play, a fair coin is tossed.

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Use x_k to denote the game position at the k^{th} step and we will make the convention that if $x_k \notin W$, then $x_\ell = x_k$ for all $\ell \geq k$.

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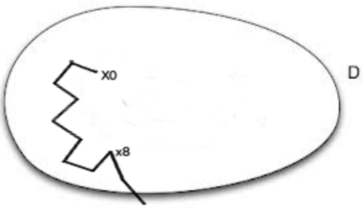
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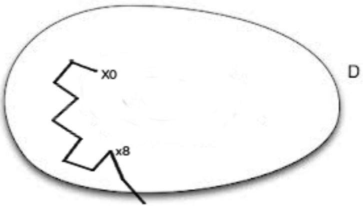

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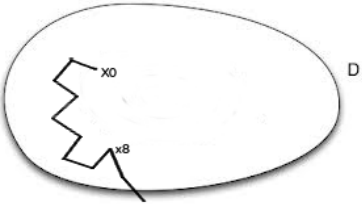

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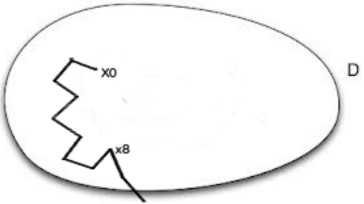

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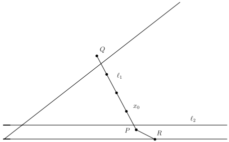

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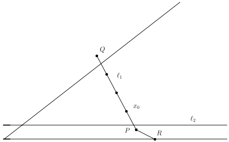

If $\pi \leq \eta < 2\pi$ then by the max-min nature of the game reduces to computing a random walk on a half line which is infinite.

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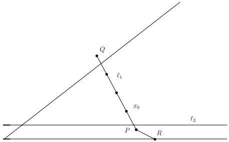

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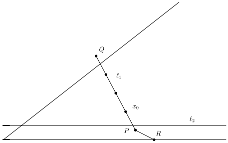

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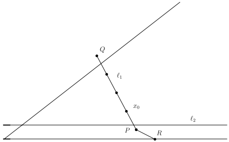

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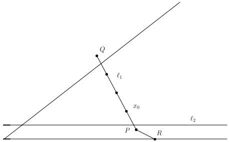

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Also ℓ_2 is the horizontal line lying $\epsilon \cos \frac{\eta}{2}$ units above the x-axis.

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So the state space of the game is a "half-tree" with trunk the x-axis and whose branches emanate at points $(k\epsilon, 0)$ on the trunk. The game position can move anywhere along a branch, but it cannot go to a lower branch.

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$$\begin{aligned} E_{N\epsilon}\left[\,\tau\,\right] &= \sum_{n=N}^{\infty} E_{N\epsilon}\left[\,\tau \,\middle|\, G_n\right] P_{N\epsilon}\left(G_n\right) \\ &= \sum_{n=N}^{\infty}\left[\sum_{j=N}^{n-1} E_{N\epsilon}\left[\,\tau_{j+1} - \tau_j \,\middle|\, G_n\right] + E_{N\epsilon}\left[\,\tau - \tau_n \,\middle|\, G_n\right]\right] P_{N\epsilon}\left(G_n\right), \end{aligned} \tag{3}$$

Lemma: For some positive A_1 and A_2, both depending only on ϵ and η, for $n \geq N+1$, with τ being the time to end the game,

$$A_1\, n^2 \leq E_{N\epsilon}\left[\, \tau - \tau_n \,\middle|\, \tau_n < \infty, \tau_{n+1} = \infty \,\right] \leq A_2\, n^2 .$$

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$$E_{N\epsilon}\left[\,\tau\,\right] < \infty \Longleftrightarrow \sum_{n=M+1}^{\infty} n^2 P_{N\epsilon}\left(G_n\right) < \infty. \tag{4}$$

Lemma: For some positive A_1 and A_2, both depending only on ϵ and η, for $n \geq N+1$, with τ being the time to end the game,

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Finally the bounds for $P_{N\epsilon}\left(G_n\right)$ are straightforward and imply Theorem 1

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$u(x) = r^2 f(\theta)$, where f is even, $f > 0$ on $\left(-\frac{\eta}{2}, \frac{\eta}{2}\right)$, $f\left(\pm\frac{\eta}{2}\right) = 0$, $f' < 0$ on $\left(0, \frac{\eta}{2}\right)$, $f' > 0$ on $\left(-\frac{\eta}{2}, 0\right)$, $f'(0) = 0$ and

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$$f \in C\left(\left[-\frac{\eta}{2}, \frac{\eta}{2}\right]\right) \cap C^{1,1/3}\left(\left[-\frac{\eta}{2}, \frac{\eta}{2}\right]\right) \cap C^\infty\left(\left[-\frac{\eta}{2}, \frac{\eta}{2}\right] \setminus \{0\}\right).$$

Theorem 2: If

$$\eta < \pi \left[1 - \frac{\sqrt{2}}{2} \right],$$

then there is $\epsilon_0 > 0$ and a strategy for player II such that the time τ to end the game in W_η satisfies

$$\sup_{S_I} \sup_{0<\epsilon<\epsilon_0} \epsilon^2 E_{x_0}[\tau] < \infty, \quad x_0 \in W_\eta,$$

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so we take G a small neighborhood of the axis and use $\Delta_\infty u = -1$ in the classical sense away from G

We can then show that for some $C_G > 0$ and sufficiently small ϵ

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So if τ is the time to end the game,

$$\frac{\epsilon^2}{2}(1 - 2C_G \, \epsilon) E_{x_0}[\tau \wedge \tau_G] \leq u(x_0).$$

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now optional stopping leads immediately to

$$\sup_{S_I} \sup_{0<\epsilon<\epsilon_1} \epsilon^2 E_{x_0}[\tau] < \infty, \quad x_0 \in W,$$

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The game A position is x_k and its ending time by τ_A. Let y_k and τ_B are for game B.

We show $\tau_B \leq \tau_A$ basic states after the n^{th} play.

State S₁(n): $\tau_B \wedge \tau_A > n$. The game positions coincide and lie within W_η: $x_n = y_n \in W_\eta$.

State S₂(n): $\tau_B \wedge \tau_A > n$. There is a closed disk $D_n \subseteq W_\eta$ such that

- (a) the radius of D_n is ϵ;
- (b) for some $m < n$, the center of D_n is y_m;
- (c) $x_n \in \partial D_n$;
- (d) $y_n \in \partial D_n$ and y_n is the point in D_n that is closest to the upper boundary of W_η.

State S₃(n): $\tau_B \wedge \tau_A > n$. There is a closed disk $D_n \subseteq W_\eta$ such that

- (a) for some $k \geq 2$, the radius of D_n is $k\epsilon$;
- (b) for some $m < n$, the center of D_n is y_m;
- (c) $x_n \in D_n$;
- (d) $y_n \in \partial D_n$ and y_n is the point in D_n that is closest to the upper boundary of W_η.

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- $y_{n+1} \notin W_\eta$ and $\tau_B \leq \tau_A$.

- Case 1 current state $S_1(n)$. If player I wins he moves the token, in both games away, from boundary and the resulting state is $S_1(n+1)$ If player II wins either $y_{n+1} \notin W_\eta$, else $S_2(n+1)$

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- Case 3. If player I wins, $S_2(n+1)$, else $S_3(n+1)$

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Neglecting technicalities, assume $u(x) = r^2 f(\theta) \geq 0$ is a solution to

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In polar coordinates it is known (Aronsson (1984)) that

$$\Delta_\infty u = \left[u_r^2 + \frac{1}{r^2} u_\theta^2 \right]^{-2} \left[u_r^2 u_{rr} + \frac{2}{r^2} u_r u_\theta u_{r\theta} + \frac{1}{r^4} u_\theta^2 u_{\theta\theta} - \frac{1}{r^3} u_r u_\theta^2 \right].$$

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$$4f^2\left[1+2f\right]+\left(f'\right)^2\left[1+6f+f''\right]=0 \text{ on } \left(-\frac{\eta}{2},\frac{\eta}{2}\right). \tag{6}$$

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It is reasonable to assume f is symmetric and concave, and this suggest $f'(0)=0$. Thus we consider (6) on $\left(0,\frac{\eta}{2}\right)$ with

$$f'(0)=0, \qquad f\left(\frac{\eta}{2}\right)=0. \tag{7}$$

Making a functional transformation to convert this to a first order boundary value problem: Let

$$y = f \text{ and } a = f(0),$$

then transform via

$$H_a(y(x)) = \left(y'(x)\right)^2. \tag{8}$$

This converts (6) into the equation

$$4y^2\left[1+2y\right] + H_a(y)\left[1+6y+\frac{1}{2}H_a'(y)\right] = 0, \tag{9}$$

and since

$$H_a(a) = H_a(y(0)) = \left(y'(0)\right)^2,$$

the condition (7) implies that

$$H_a(a) = 0. \tag{10}$$

We get an implicit representation of $y = f(\theta)$:

$$\theta = \int_{y(\theta)}^{a} \frac{dw}{\sqrt{H_a(w)}}, \qquad \theta \in \left(0, \frac{\eta}{2}\right).$$

In particular,

$$\begin{aligned} \frac{\eta}{2} &= \int_{y(\eta/2)}^{a} \frac{dw}{\sqrt{H_a(w)}} \\ &= \int_{0}^{a} \frac{dw}{\sqrt{H_a(w)}}. \end{aligned} \tag{11}$$

We are able to show that the boundary value problem (9)–(10) is solvable on $[0, a]$ for any $a > 0$

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To evaluate this limit, it is necessary to make one more change of variables:

$$G_a(y) = (ay)^{-2} H_a(ay), \qquad 0 < y < 1.$$

Thanks for a great FPD!