Random Walks in a Sparse Random Environment

Youngsoo Seol

Texas State University-San Marcos
Department of Mathematics

May 18, 2014

Topic of the Research: RWSRE on $\mathbb{Z}$

- We introduce *Random Walks in a Sparse Random Environment* (RWSRE) on $\mathbb{Z}$
- We study their asymptotic behavior.
- RWSRE is a cousin of *Random Walks in Random Environment* (RWRE)
- RWRE on $\mathbb{Z}$ is an active research area but their basic properties are well-understood by now.
- The new model is well-motivated:
  1. RWSRE combines features of several existing models of *random motion in random media* and admits a transparent physics interpretation.
  2. RWSRE exhibits a richer spectrum of asymptotic regimes than RWRE

■ ***RWSRE appears to be a simple, elegant, natural, flexible, and yet complex and challenging mathematically generalization RWRE***

Outline of the Talk

1. **Model. RWSRE on $\mathbb{Z}$**
2. **Basic asymptotic behavior: Recurrence/Transience and asymptotic speed**
3. **Limit theorem in a recurrent regime**

Relevant results for RWRE will be sketched in the talk.

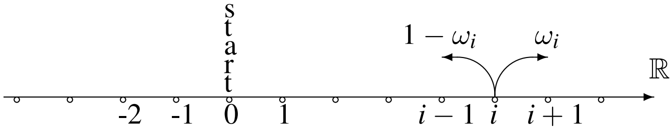
Random Walks in Random Environments on $\mathbb{Z}$

Random Environment

- An environment $\omega = (\omega_x)_{x\in\mathbb{Z}}$ is an element of $\Omega = [0,1]^{\mathbb{Z}}$.
- ω_x serves to determine transition probabilities at $x \in \mathbb{Z}$.
- The distribution of ω is denoted by P.
- P is a stationary and ergodic measure on Ω.

Random Walk in a Fixed Environment (Quenched Law P_ω)

- For a fixed ω, the random walk X_n is a Markov chain on $\mathbb{Z}$.
- $P_\omega(X_0 = 0) = 1$ and transition probabilities are given by
  $P_\omega(X_{n+1} = i+1 | X_n = i) = \omega_i$,
  $P_\omega(X_{n+1} = i-1 | X_n = i) = 1 - \omega_i$.



RWRE: Annealed Setting

Annealed (Average) Law $\mathbb{P}$ for RWRE

- By averaging P_ω with respect to the environment distribution P :

$$\mathbb{P}(A) = \int_\Omega P_\omega(A)P(d\omega) = E_P(P_\omega(A))$$

for measurable events A.

Summary and Notations:

- P is the law of the ***random environment***, and the expectation under P is denoted by E_P.
- For a fixed environment ω, P_ω is the law of the ***random walk***, and the expectation under P_ω is denoted by E_ω.
- $\mathbb{P}$ is the law of ***RWRE***, and the expectation under $\mathbb{P}$ is denoted by $\mathbb{E}$.
- The random walk X_n is a Markov chain under P_ω, but not under $\mathbb{P}$.

Random Potential Associated with RWRE

$$\rho_n = \frac{1-\omega_n}{\omega_n}, \qquad n \in \mathbb{Z}.$$

It turns out that asymptotic results for one-dimensional RWRE can usually be stated in terms of certain averages of functions of ρ_0 and explained by means of typical “landscape features" (such as *traps* and *valleys*) of the *random potential* $(V_n)_{n\in\mathbb{Z}}$, which is associated with the random environment as follows: $V_0 = 0$ and

$$V_n = \begin{cases} \sum_{k=1}^{n} \log \rho_k & \text{if } n > 0, \\ -\sum_{k=1}^{|n|} \log \rho_{-k} & \text{if } n < 0. \end{cases}$$

Exit Probabilities of RWRE

- ***An illustration of the important paradigm: RWRE on $\mathbb{Z}$ is a "random motion in a random potential"***

Let $T_n = \inf\{k \geq 0 : X_k = n\}$. For any $m_-, m_+ \in \mathbb{N}$ and an integer $z \in [-m_-, m_+]$, define

$$\begin{aligned} \phi_{\omega,z}(m_-, m_+) &:= P_\omega^z(\{X_n\} \text{ hits } -m_- \text{ before hitting } m_+) \\ &= P_\omega^z(T_{-m_-} < T_{m_+}). \end{aligned}$$

Due to the Markov Property, $\phi_{\omega,z}(a_-, a_+)$ as a function of z is harmonic function for the random walk:

$$\phi_{\omega,z}(m_-, m_+) = (1 - \omega_z)\phi_{\omega,z-1}(m_-, m_+) + \omega_z \phi_{\omega,z+1}(m_-, m_+)$$

with boundary conditions $\phi_{\omega,-m_-}(m_-, m_+) = 1$ and $\phi_{\omega,m_+}(m_-, m_+) = 0$.

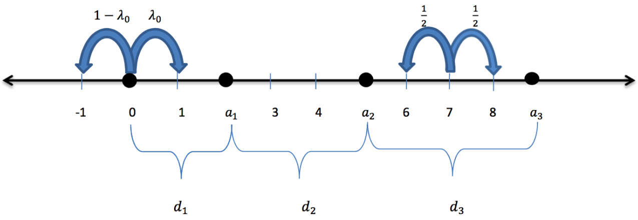
$$\phi_{\omega,z}(m_-, m_+) = \frac{\sum_{i=z+1}^{m_+} \prod_{j=z+1}^{i-1} \rho_j}{\sum_{i=z+1}^{m_+} \prod_{j=z+1}^{i-1} \rho_j + \sum_{i=-m_-+1}^{z} \prod_{j=i}^{z} \rho_j^{-1}}.$$

RWSRE on $\mathbb{Z}$.
General Idea: A perturbation of the Simple Random Walk

- ***Random Walk in a Random Non-Homogeneous Media***

$\mathcal{A} = (a_n)_{n\in\mathbb{Z}}$ locations of *impurities* in a random media

$$\omega_n = \begin{cases} \lambda_k & \text{if } n = a_k \text{ for some } k \in \mathbb{Z}, \\ 1/2 & \text{otherwise.} \end{cases}$$



To be specified: $d_n = a_n - a_{n-1}$ and λ_n

Sparse Random Environment: Rigorous Treatment

For the Purpose of the Talk

- Let $(d_k)_{k\in\mathbb{Z}}$ be a sequence of i.i.d. strictly positive integer-valued random variables. For $n \in \mathbb{Z}$ let

$$a_n = \begin{cases} \sum_{k=1}^{n} d_k & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{k=-n}^{-1} d_k & \text{if } n < 0. \end{cases}$$

  Thus $(a_n)_{n\in\mathbb{Z}}$ is a *two-sided renewal sequence*.
- Let $(\lambda_n)_{n\in\mathbb{Z}}$ is an i.i.d. sequence *independent of* $(d_k)_{k\in\mathbb{Z}}$.

More Generally

- $(d_n, \lambda_n)_{n\in\mathbb{Z}}$ is stationary and ergodic.
- Thus $(a_n, \lambda_n)_{n\in\mathbb{Z}}$ is a *marked point process* on $\mathbb{Z} \times [0, 1]$.
  Marked sites: a_n, Marks: λ_n, Point process: $N(A) = \sum_{n\in\mathbb{Z}} \mathbf{1}_{\{a_n \in A\}}$.

RWRE on $\mathbb{Z}$: Recurrence/Transience

- *Recurrent*: Infinitely many visits to any point.
- *Transient*: Finitely many visits to any point (possibly zero).

- A crucial statistic is:

$$\rho_x = \frac{1-\omega_x}{\omega_x}$$

Whether we have transience or recurrence is determined by $E_P(\log \rho_0)$:

Theorem (Solomon 75, Alili 99)

1. *If* $E_P(\log \rho_0) < 0$, *then* $\lim_{n\to\infty} X_n = +\infty$, $\mathbb{P}$ – a. s.
2. *If* $E_P(\log \rho_0) > 0$, *then* $\lim_{n\to\infty} X_n = -\infty$, $\mathbb{P}$ – a. s.
3. *If* $E_P(\log \rho_0) = 0$, *then* X_n *is recurrent* $\mathbb{P}$ – a. s.

RWSRE on $\mathbb{Z}$: Recurrence and Transience

Let $\xi_i = \frac{1-\lambda_i}{\lambda_i}$.

Theorem

Suppose that the following three conditions are satisfied:

1. *The sequence of pairs $(d_n, \lambda_n)_{n\in\mathbb{Z}}$ is stationary and ergodic*
2. $E_P(\log \xi_0)$ *exists (possibly infinite)*
3. $E_P\big(\log d_0\big)$ *exists and is finite.*

Then:

(a) $E_P(\log \xi_0) < 0$ *implies* $\lim_{n\to\infty} X_n = +\infty$, $\mathbb{P} - \text{a. s.}$

(b) $E_P(\log \xi_0) > 0$, *implies* $\lim_{n\to\infty} X_n = -\infty$, $\mathbb{P} - \text{a. s.}$

(c) $E_P(\log \xi_0) = 0$ *implies that* X_n *is recurrent,* $\mathbb{P} - \text{a. s.}$

As long as $E_P\big(\log d_0\big) < \infty$, the sparse environment ω induces the same recurrence-transience behavior as the underlying random environment λ.

RWSRE on $\mathbb{Z}$: Recurrence and Transience

Theorem

Suppose that the following conditions hold:

1. *The sequence of pairs $(d_n, \lambda_n)_{n\in\mathbb{Z}}$ is stationary and ergodic*
2. *The random variables d_n are i.i.d.*
3. $E_P\big(|\log \xi_0|\big) < +\infty$ *while* $E_P(\log d_0) = +\infty$.

Then, $\liminf_{n\to\infty} X_n = -\infty$ *and* $\limsup_{n\to\infty} X_n = +\infty$, $\mathbb{P}-\text{a.s.}$

This theorem shows that the opposite phenomenon(namely, the property of λ are essentially irrelevant to the basic asymptotic behavior of X_n) occur when $E_P\big(\log d_0\big) = \infty$.

RWRE on $\mathbb{Z}$: Law of the Large Numbers

$$\overline{S} = 1 + 2\sum_{i=0}^{\infty}\prod_{j=0}^{i}\rho_j, \qquad \overline{F} = 1 + 2\sum_{i=1}^{\infty}\prod_{j=0}^{i-1}\rho_{-j}^{-1}$$

Theorem (Solomon 75, Molchanov 91, Alili 99, Zeitouni 04)

1. *If* $E_P(\overline{S}) < \infty$, *then* $\lim_{n\to\infty}\frac{X_n}{n} = \frac{1}{E_P(\overline{S})}$.
2. *If* $E_P(\overline{F}) < \infty$, *then* $\lim_{n\to\infty}\frac{X_n}{n} = -\frac{1}{E_P(\overline{F})}$.
3. *If* $E_P(\overline{S}) = \infty$ *and* $E_P(\overline{F}) = \infty$, *then* $\lim_{n\to\infty}\frac{X_n}{n} = 0$.

Theorem (For i.i.d. environments:)

1. *If* $E_P(\rho_0) < 1$, *then* $\lim_{n\to\infty}\frac{X_n}{n} = \frac{1-E_P(\rho_0)}{1+E_P(\rho_0)}$.
2. *If* $E_P(\rho_0^{-1}) < 1$, *then* $\lim_{n\to\infty}\frac{X_n}{n} = -\frac{1-E_P(\rho_0^{-1})}{1+E_P(\rho_0^{-1})}$.
3. *If* $E_P(\rho_0)^{-1} \le 1 \le E_P(\rho_0^{-1})$, *then* $\lim_{n\to\infty}\frac{X_n}{n} = 0$.

RWSRE on $\mathbb{Z}$: Law of the Large Numbers

$$\overline{S}_\lambda = 1 + 2\sum_{i=0}^{\infty}\prod_{j=0}^{i}\xi_j \quad \text{and} \quad \overline{F}_\lambda = 1 + 2\sum_{i=1}^{\infty}\prod_{j=0}^{i-1}\xi_{-j}^{-1}.$$

Theorem

Let the conditions of Theorem for recurrent and transience hold. Suppose in addition that $(d_n)_{n\in\mathbb{Z}}$ *is independent of* $(\lambda_n)_{n\in\mathbb{Z}}$ *under* P*. Then the asymptotic speed of the RWSRE exists* $\mathbb{P}-\text{a. s.}$ *Moreover,*

$$\mathbb{P}\big(\lim_{n\to\infty} X_n/n = \mathrm{v}_P\big) = \mathbb{P}\big(\lim_{n\to\infty} T_n/n = 1/\mathrm{v}_P\big) = 1,$$

where $\mathrm{v}_P \in (-1,1)$ *is a constant whose reciprocal* v_P^{-1} *as the followings:*

1. *If* $\lim_{n\to\infty} X_n = +\infty$*, then* $\frac{1}{\mathrm{v}_P} = \frac{\mathrm{VAR}_P(d_0)}{E_P(d_0)} + E_P(\overline{S}_\lambda)\cdot E_P(d_0).$
2. *If* $\lim_{n\to\infty} X_n = -\infty$*, then* $\frac{1}{\mathrm{v}_P} = -\frac{\mathrm{VAR}_P(d_0)}{E_P(d_0)} - E_P(\overline{F}_\lambda)\cdot E_P(d_0).$

RWSRE on $\mathbb{Z}$: Proof of the Law of Large Numbers

- ***Hitting time decomposition***:

▶ *Decomposition for T_{a_1}*:

$$T_{a_1} = 1 + \mathbf{1}_{\{X_1=1\}}[\mathbf{1}_{\{\widetilde{T}_0<\widetilde{T}_{a_1}\}}(\widetilde{T}_0 + T'_{a_1}) + \mathbf{1}_{\{\widetilde{T}_0>\widetilde{T}_{a_1}\}}\widetilde{T}_{a_1}]$$
$$+\mathbf{1}_{\{X_1=-1\}}[\mathbf{1}_{\{\widehat{T}_0<\widehat{T}_{a_{-1}}\}}(\widehat{T}_0 + T''_{a_1}) + \mathbf{1}_{\{\widehat{T}_0>\widehat{T}_{a_{-1}}\}}(\widehat{T}_{a_{-1}} + T'_0 + T'''_{a_1})],$$

where

$$\widetilde{T}_0 = \inf\{n > T_1 : X_n = 0\}, \qquad \widetilde{T}_0 + T'_{a_1} = \inf\{n > \widetilde{T}_0 : X_n = a_1\},$$
$$\widetilde{T}_{a_1} = \inf\{n > T_1 : X_n = a_1\}, \qquad \widehat{T}_0 = \inf\{n > T_{-1} : X_n = 0\},$$
$$\widehat{T}_0 + T''_{a_1} = \inf\{n > \widehat{T}_0 : X_n = a_1\}, \qquad \widehat{T}_{a_{-1}} = \inf\{n > T_{-1} : X_n = a_{-1}\},$$
$$\widehat{T}_{a_{-1}} + T'_0 = \inf\{n > \widehat{T}_{a_{-1}} : X_n = 0\},$$

$$\widehat{T}_{a_{-1}} + T'_0 + T'''_{a_1} = \inf\{n > T_{a_{-1}} + T'_0 : X_n = a_1\}.$$

RWSRE on $\mathbb{Z}$: Proof of the Law of Large Numbers

- ***Our strategy***:

To do end we show the following lemmas.

- *Lemma 1*: τ_{a_i} is stationary and ergodic under $\mathbb{P}$.
- *Lemma 2*:
  (a) $\mathbb{E}(T_{a_1}) = \text{VAR}_P(d_1) + E_P(\overline{S}_\lambda) \cdot [E_P(d_1)]^2$,
  (b) $\mathbb{E}(T_{a_{-1}}) = \text{VAR}_P(d_1) + E_P(\overline{F}_\lambda) \cdot [E_P(d_1)]^2$.
- *Lemma 3*: suppose that $\lim_{n\to\infty} \frac{T_{a_n}}{n} = \alpha$, $\mathbb{P}$ − a. s., for some constant $\alpha \leq \infty$. Then,

$$\lim_{n\to\infty} \frac{T_n}{n} = \frac{\alpha}{E_P(d_1)} \quad \text{and} \quad \lim_{n\to\infty} \frac{X_n}{n} = \frac{E_P(d_1)}{\alpha}, \quad \mathbb{P} - \text{a. s.}$$

- *Lemma 4*: $\frac{T_n}{n} = \frac{\sum_{i=1}^{n} \tau_{a_i}}{n} \to \mathbb{E}(T_{a_1})$.

Classical result: Limit Theorem for Sinai's RWRE

Assumptions

1. $E_P(\log \rho_0) = 0$ *(recurrence)*
2. $\sigma_P^2 := E_P(\log \rho_0)^2 \in (0, \infty)$.
3. *Recall the random potential*

$$V_n = V_n(\omega) = \begin{cases} \sum_{i=1}^{n} \log \rho_i & \text{if } n > 0, \\ -\sum_{i=n}^{-1} \log \rho_i & \text{if } n < 0. \end{cases}$$

Then $\left\{ \frac{V_{[nt]}}{\sigma_P \sqrt{n}} \right\}_{t \in \mathbb{R}} \Rightarrow$ *Brownian Motion.*

Theorem (Sinai 82)

$\frac{X_n}{(\log n)^2} \Rightarrow \sigma_P b_\infty$, *where* b_∞ *is the "location of the bottom of the deepest valley" of a Brownian motion* $\{B_t\}_{t \in [0,1]}$.

Limit Theorem for RWSRE in Sinai's Regime

Assumptions

1. $E_P(\log \rho_0) = 0$ *(recurrence)*
2. $\sigma_P^2 := E_P(\log \rho_0)^2 \in (0, \infty)$ *and* $\frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \log \rho_k \Rightarrow B(t)$.
3. $P(d_1 > t) \sim t^{-\alpha} h(t)$ *where* $\alpha \in (0, 1)$ *and* $h(t)$ *is slowly varying. In particular, we have:* $U_n(t) := \frac{1}{r_n} \sum_{k=1}^{[nt]} d_k \Rightarrow G_\alpha(t)$ *where* $G_\alpha(t)$ *is an asymmetric stable process and* $r_n \sim n^{\frac{1}{\alpha}} h_1(n)$,

Theorem

$\frac{X_n}{u(\log n)} \Rightarrow b_\infty$, *where* $u(n) = (n)^{\frac{2}{\alpha}} h_2(n)$.

Lemma (Main auxiliary lemma)

$\widehat{R}_n(t) := \frac{1}{\log n} \sum_{k=1}^{\lfloor u(\log n)t \rfloor} \log \rho_k \Rightarrow V_\alpha$, *for some sequence* $u(n) \in \mathcal{R}_{2/\alpha}$.

Unusual Scaling

- Random variable b_∞ in our theorem differ from their prototypes in Sinai's theorem. b_∞ in our model is defined in terms of the process V_α which is a symmetric Lévy process but b_∞ in sinai model is related to the valley of Brownian motion.
- The magnitude order $(\log n)^2$ of Sinai's random walk is much smaller than $n^{\frac{1}{2}}$,which is the magnitude order of the simple symmetric random walk.
- Sinai's scaling $(\log n)^2$ for the location of the random walk after n steps is generalized to basically $(\log n)^\alpha$, with $\alpha > 0$ being a parameter determined by the distribution of the distance between two successive impurities of the media.
- Extremely sparse impurities in the media have a dramatic impact on the motion.(It has essentially the same impact on the motion as the usual random environment in sinai's regime. The magnitude order $n^{\frac{1}{2}}$ of the simple symmetric random walk is transformed to the magnitude order $(\log n)^\alpha$ of our model.)

Valleys of a Scaled Potential for Sparse Environment

Define the normalized random potential associated with the sparse environment as the following:

- *Define:*

$$\widehat{R}_n(t) = \mathrm{sign}(t) \cdot \frac{1}{\log n} \sum_{k=1}^{[u(\log n)t]} \log \rho_k = \mathrm{sign}(t) \cdot \frac{1}{\log n} \sum_{k=1}^{\eta[u(\log n)t]} \log \xi_k.$$

  Then $\{\widehat{R}_n(t) : t \geq 0\}$ converges weakly in $D(\mathbb{R}_+, R)$ to the process V_α.
- *Definition:* A triple $(B^{'}, b, B^{''})$ is called a valley of the path if

$$\hat{R}_n(b) = \min_{B^{'} \leq t \leq B^{''}} \hat{R}_n(t), \quad \hat{R}_n(B^{'}) = \max_{B^{'} \leq t \leq b} \hat{R}_n(t),$$

$$\hat{R}_n(B^{''}) = \max_{b \leq t \leq B^{''}} \hat{R}_n(t).$$

- *Definition:* The depth of the valley is defined as

$$d[B^{'}, b, B^{''}] := \min\{\hat{R}_n(B^{'}) - \hat{R}_n(b), \hat{R}_n(B^{''}) - \hat{R}_n(b)\}.$$

RWSRE: Rescaled Random Potential

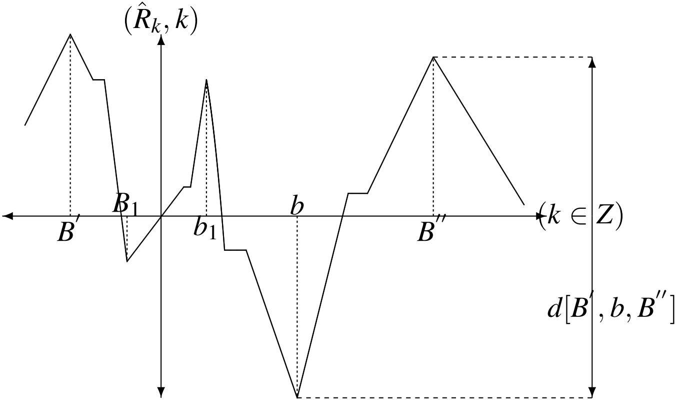


Figure 5.2. A rescaled valley and refinement operation

Heuristic of Limit Theorem for Recurrent Regime

- Let b_n be the bottom of the deepest valley of $\widehat{R}_n(t)$. Write:

$$P_\omega\left(\left|\frac{X_n}{u(\log n)} - b_n\right| > \epsilon\right) \to_{n\to\infty} 0$$

- Because of the the scaling limit (to a LP), the deepest valley of

$$\widehat{R}_n(t) = \operatorname{sign}(t) \cdot \sum_{i=1}^{[t]} \log \rho_i, \quad 0 \le t \le u(\log n),$$

  is $\sim \log n$. It takes $\sim e^L$ units of time to pass a barrier of height L. Hence, it takes $\sim e^{\log n} = n$ to escape from the deepest valley.

- Therefore, the RWSRE spends all the time at the bottom of the deepest valley! Since $\widehat{R}_n(t) \Rightarrow V_\alpha$, the deepest valley (of the depth $\sim \log n$) is located at the distance of order $u(\log n)$ from the origin.

THANK YOU!

Main References

- Solomon, F. *Random walks in a random environment.* Ann. Probab. **3** (1975), 1–31.
- Sinai, Ya. G. *The limiting behavior of a one-dimensional random walk in a random medium.* Th. Probab. Appl. **27** (1982), 256-268.
- Zeitouni, O. *Random Walks in Random Environment.* In Lectures on probability theory and statistics, volume 1837 of Lecture Notes in Math. (2004), 189–312, Springer, Berlin.
- Kesten, H. *The limit distribution of sinai's random walk in random environment.* Physica. A **138** (1986), 299–309
- Golosov, A.O. *On limiting distributions for a random walk in a critical one-dimensional random environment.* Russian Math. Surveys. **41** (1986), 199–200.