

The Growth Model: Busemann Functions, Shape, Geodesics, and Other Stories

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Summary/Conclusion

We consider the growth model (oriented last passage percolation) on the two-dimensional square lattice.

Want to describe the large-scale behavior of the model.

In the special case of exponential or geometric weights, explicit computations are possible and many results exist.

Our long-term goal is to develop an approach for general weight distribution.

Busemann functions play a central role: boundary weights for stationary processes (like invariant measures are for Markov chains), infinite geodesics, shape formula, solve variational formulas, tool for proving results about fluctuation exponents.

Summary/Conclusion

In geometry: Busemann functions are used to study large-scale geometry in metric spaces.

In first passage percolation (random geometry on \mathbb{Z}^2): introduced by Newman in mid '90ies then Hoffman in '08 to study infinite geodesics.

Licea-Newman '96 proved almost-sure existence of Busemanns by first proving coalescence of directional geodesics, under a global (unproven) curvature assumption.

Recently Damron-Hanson used a weak limit to get a version of Busemann functions and show existence of directional infinite geodesics in first passage percolation under weaker assumptions than those of Newman: differentiability or strict convexity of the shape is enough.

For the growth model, we prove **almost sure** existence of Busemann functions, then describe the shape function and geodesics under similar assumptions to those of Damron-Hanson.

Last Passage Percolation

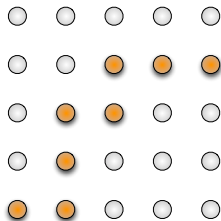
Random potential $\omega = (\omega_x)_{x \in \mathbb{Z}^2} \in \Omega$, \mathbb{R} -valued i.i.d., $2 + \varepsilon$ moments.

Think of ω_x as time spent at x (even though they can be negative).

Up-right paths $x_{0,n} = (x_0, \dots, x_n)$

take steps $e_1 = (1, 0)$ or $e_2 = (0, 1)$.

Passage time of path $x_{0,n}$ is $\sum_{i=0}^{n-1} \omega_{x_i}$.



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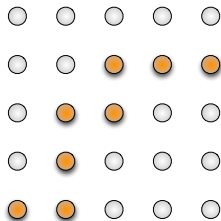
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Point-to-point last passage time: $G_{x,y}(\omega) = \max_{\substack{x_{0,n} \\ x_0=x, x_n=y}} \sum_{k=0}^{n-1} \omega_{x_k}$.

Point-to-level last passage time: $G_{(n)}(\omega) = \max_{x_{0,n}: x_0=0} \sum_{k=0}^{n-1} \omega_{x_k}$.

Shape Theorem

LLN says sum of i.i.d. grows linearly.

$G_{0,x}$ is not quite a sum of i.i.d.

It is however superadditive: $G_{x,y} + G_{y,z} \leq G_{x,z}$.

Then: outside one null set, for each $\xi \in \mathbb{R}_+^2$

$g_{pp}(\xi) = \lim_{n \rightarrow \infty} n^{-1} G_{0,[n\xi]}$ exists almost-surely, is concave, homogenous
($g_{pp}(c\xi) = cg_{pp}(\xi)$) and continuous all the way to the boundary.

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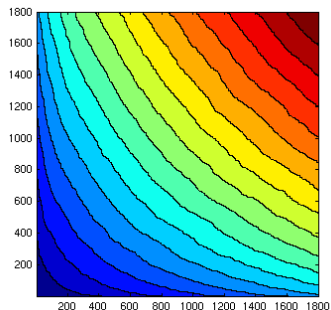
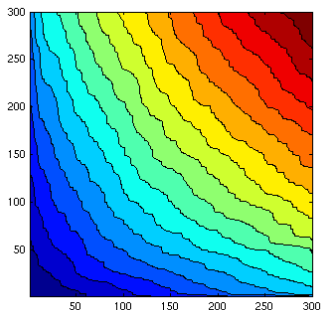
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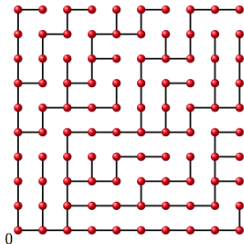
Geodesics

Path $x_{0,n}$ that maximizes $G_{x,y}$ is called a **geodesic**.

Unique if ω_0 is continuous. Otherwise, can talk e.g. about topmost and rightmost geodesics.

This way we have a tree of geodesics from 0.

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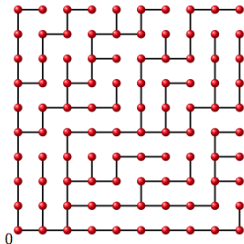
By compactness, at least one infinite geodesic exists.

But is there an infinite geodesic in any given direction $\xi \in \mathcal{U} = \{(t, 1-t) : t \in (0, 1)\}$?

Is it the limit of the geodesic from 0 to $[n\xi]$ as $n \rightarrow \infty$?

If ω_0 is continuous, are these **ξ -geodesics** unique?

Do ξ -geodesics out of x and y coalesce (i.e. eventually merge)?



Licea and Newman '96: answers are in the positive for standard first passage percolation (nearest-neighbor paths minimizing the passage time) if $g_{pp}(\xi)$ satisfies a global curvature assumption.

Problem: the curvature assumption has not been proved. Though conjectured to hold.

Damron and Hanson '14: Existence holds under just strict convexity or differentiability of g_{pp} (which should be “easier” to prove).

Ferrari and Pimentel '05: answers are in the positive also for the last passage percolation model we are considering, but with ω_0 exponential.

The exponential model is one of the **solvable** models for which explicit computations are possible. In particular, an explicit formula is available for the shape $g_{pp}(\xi)$.

Would like to allow more general weight distributions.

Fluctuations

CLT says that if $X_{0,n}$ has increments e_1 or e_2 equally likely, then it fluctuates from its average (straight line from 0 to $(n/2, n/2)$) by $n^{1/2}$.

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What are the fluctuations of the geodesic from 0 to $[n\xi]$?

What about geodesic from 0 to level n ? (i.e. the path maximizing $G_{(n)}$)

Conjecture: with enough moments on ω_0 (Auffinger and Louidor '11) geodesic fluctuations are of order $n^{2/3}$.

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On the other hand, $G_{0,[n\xi]}$ (and $G_{(n)}$) should have $n^{1/3}$ fluctuations.

Limit distributions related to **Tracy-Widom** from random matrices.

Models with these fluctuation exponents are said to belong to the **Kardar-Parisi-Zhang (KPZ)** universality class.

Johansson '00 proved LPP with exponential weights is in the KPZ class.

Again: solvability of the model was key.

Understanding the shape

Consider a finite subset $V \subset \mathbb{Z}^2$ containing 0 (e.g. $\{u : |u| \leq L\}$).

$\{G_{0,[n\xi]-u} - G_{0,[n\xi]} : u \in V\}$ describes the **microscopic** shape around $[n\xi]$.

We expect this random vector to converge in distribution as $n \rightarrow \infty$.

Shifting by $-[n\xi]$ and reflecting $\omega_x \mapsto \omega_{-x}$ turns the above into

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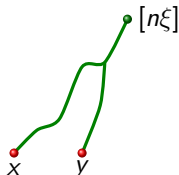
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Busemann functions: $B^\xi(x, y; \omega) = \lim_{n \rightarrow \infty} (G_{x,[n\xi]} - G_{y,[n\xi]}).$

Limit exists if e.g. geodesics coalesce.



Understanding geodesics

Note that $G_{x,[n\xi]} = \omega_x + \max(G_{x+e_1,[n\xi]}, G_{x+e_2,[n\xi]})$.

So $(G_{x,[n\xi]} - G_{x+e_1,[n\xi]}) \wedge (G_{x,[n\xi]} - G_{x+e_2,[n\xi]}) = \omega_x$ almost surely.

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$n \rightarrow \infty$ gives $B^\xi(x, x + e_1) \wedge B^\xi(x, x + e_2) = \omega_x$ almost surely.

The above suggests that ξ -geodesic out of x should follow the smallest $B^\xi(x, x + z)$, $z \in \{e_1, e_2\}$.

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If a tie occurs, then can go either way:

Always breaking ties with e_1 should give the rightmost ξ -geodesic

Always breaking ties with e_2 should give the topmost ξ -geodesic

Busemann functions

Consider g_{pp} as a concave function on $\mathcal{U} = \{(t, 1 - t) : t \in (0, 1)\}$.

Given $\xi \in \mathcal{U}$ at which g_{pp} is differentiable, let $[\underline{\xi}, \bar{\xi}] \subset \mathcal{U}$ be the maximal (possibly degenerate) interval containing ξ on which g_{pp} is linear.

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Standing assumptions: $\mathbb{P}\{\omega_0 \geq c\} = 1$, ω_x i.i.d. with $2 + \varepsilon$ moments, $\xi \in \mathcal{U}$, and $\underline{\xi}$ and $\bar{\xi}$ are points of differentiability.

Theorem. $B^\xi(x, y; \omega) = \lim_{n \rightarrow \infty} (G_{x, [n\underline{\xi}]} - G_{y, [n\underline{\xi}]})$ exists almost surely.

Furthermore: Same limit for all directions in the same linear segment.

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Remark. $\omega_0 \geq c$ only because we use results from queuing where service times were assumed nonnegative. All the queuing results seem to go through without this assumption.

Properties

L^1 : $\mathbb{E}[|B^\xi(x, y)|] < \infty$.

Stationary: $B^\xi(x, y; T_z\omega) = B^\xi(x + z, y + z; \omega)$ ($(T_z\omega)_x = \omega_{x+z}$)

Cocycle: $B^\xi(x, y) + B^\xi(y, z) = B^\xi(x, z)$.

The space of L^1 stationary cocycles: \mathcal{C} .

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Potential recovery: $B^\xi(0, e_1) \wedge B^\xi(0, e_2) = \omega_0$ almost surely.

Monotonicity: $\xi \cdot e_1 > \zeta \cdot e_1$ (ξ to the right of ζ) \Rightarrow

$$B^\xi(0, e_1) \leq B^\zeta(0, e_1) \text{ and } B^\xi(0, e_2) \geq B^\zeta(0, e_2)$$

$(G_{0, [n\xi]} - G_{e_1, [n\xi]} \leq G_{0, [n\zeta]} - G_{e_1, [n\zeta]})$ is forced by planarity.)

(One consequence will be: ξ -geodesic is to the right of ζ -geodesic.)

If $B \in \mathcal{C}$ then a B -geodesic is a path that follows the minimal $B(x, x+z)$, $z \in \{e_1, e_2\}$. (In case of ties, it can go either way.)

Theorem. If B recovers potential ω ($B(0, e_1) \wedge B(0, e_2) = \omega_0$ a.s.) then a B -geodesic is a geodesic: every finite piece of it is a geodesic.

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Given ξ where g_{pp} is differentiable, recall $[\underline{\xi}, \bar{\xi}]$. A geodesic $x_{0,\infty}$ is directed in $[\underline{\xi}, \bar{\xi}]$ if all limit points of x_n/n belong to this interval.

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Theorem.

- Any B^ξ -geodesic is directed in $[\underline{\xi}, \bar{\xi}]$.
- Any geodesic directed in $[\underline{\xi}, \bar{\xi}]$ is a B^ξ -geodesic.
- The B^ξ -geodesic with e_2 -tie breaks is the topmost of all geodesics directed in $[\underline{\xi}, \bar{\xi}]$. Similarly for rightmost.

Corollary. If g_{pp} is differentiable everywhere, then every geodesic is directed in $[\underline{\xi}, \bar{\xi}]$ for some ξ .

Remark. Can also handle corners using geodesics ordering, but will omit.

Thus can show: If g_{pp} is strictly concave, then every geodesic has a direction ξ , i.e. $\lim x_n/n$ exists almost surely.

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Thus can show: If g_{pp} is strictly concave, then every geodesic has a direction ξ , i.e. $\lim x_n/n$ exists almost surely.

Theorem. Assume also $\mathbb{P}\{\omega_0 \leq r\}$ is continuous in r . Then $\mathbb{P}\{B^\xi(0, e_1) = B^\xi(0, e_2)\} = 0$.

Corollary. If ω_0 is continuous, then there exists a unique geodesic directed in $[\underline{\xi}, \bar{\xi}]$ out of every point $x \in \mathbb{Z}^2$.

Theorem. Topmost $[\underline{\xi}, \bar{\xi}]$ -directed geodesics coalesce, rightmost $[\underline{\xi}, \bar{\xi}]$ -geodesics coalesce, and when ω_0 is continuous, $[\underline{\xi}, \bar{\xi}]$ -geodesics coalesce.

Variational formula

Until recently, the only description of $g_{\text{pp}}(\xi)$ was from superadditivity:

$$g_{\text{pp}}(\xi) = \sup_n n^{-1} \mathbb{E}[G_{0, [n\xi]}].$$

Going through random polymer models:

Theorem. $g_{\text{pp}}(\xi) = \inf_{B \in \mathcal{C}} \text{ess sup} \{ \omega_0 - B(0, e_1; \omega) \wedge B(0, e_2; \omega) - h(B) \cdot \xi \}.$

(Recall: \mathcal{C} is class of L^1 stationary cocycles, $\mathbb{E}[B(x, y)] = h(B) \cdot (x - y)$.)

This is a similar formula to the one in Arjun Krishnan's earlier talk on first passage percolation.

Such formulas are important in statistical mechanics: their solutions are expected to describe the infinite-volume system (i.e. geodesics and shape as $n \rightarrow \infty$).

Theorem. Under the standing assumptions, B^ξ solves the variational formula for $g_{pp}(\xi)$. In fact, the essential supremum is not needed and we have almost surely

$$g_{pp}(\xi) = \omega_0 - B^\xi(0, e_1, \omega) \wedge B^\xi(0, e_2, \omega) - h(B^\xi) \cdot \xi.$$

(In homogenization theory, a solution like B^ξ that removes the need for ess sup is called a **corrector**.)

Corollary. Due to potential recovery, we have $g_{pp}(\xi) = -h(B^\xi) \cdot \xi$.

Using some calculus one then gets that $h(B^\xi) = -\nabla g_{pp}(\xi)$.

Nice interpretation: average **microscopic** gradient is **macroscopic** gradient.

Solvable models

When ω_0 are exponential or geometric we in fact can calculate explicitly the distributions of $B^\xi(0, e_1)$ and $B^\xi(0, e_2)$ for all $\xi = (\xi_1, \xi_2)$.

For example, if ω_0 is exponential with rate 1, then $B^\xi(0, e_1)$ is exponential with rate α and $B^\xi(0, e_2)$ is exponential with rate $1 - \alpha$, where

$$\alpha = \frac{\sqrt{\xi_1}}{\sqrt{\xi_1} + \sqrt{\xi_2}}.$$

Then $h(B^\xi) = -(\mathbb{E}[B^\xi(0, e_1)], \mathbb{E}[B^\xi(0, e_2)]) = -(\frac{1}{\alpha}, \frac{1}{1-\alpha})$

and $g_{pp}(\xi) = -h(B^\xi) \cdot \xi = \xi_1 + \xi_2 + 2\sqrt{\xi_1\xi_2}$.

This is the known formula derived by Rost '81.

Fluctuations

When ω_0 is exponential or geometric, $B^\xi(ne_1, (n+1)e_1)$ are i.i.d. and so are $B^\xi(ne_2, (n+1)e_2)$.

Balázs, Cator, and Seppäläinen '06 used this to prove the $n^{2/3}$ fluctuations of the geodesic and $n^{1/3}$ fluctuations of the last passage time, in the exponential weights case, with less technology than Johansson's proof of the Tracy-Widom limit.

More generally, CLT exponents for fluctuations of $B^\xi(0, ne_1)$ and $B^\xi(0, ne_2)$ imply information about fluctuation exponents of last passage quantities. (The above BCS result is one way to achieve this.)

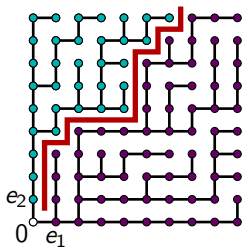
Now we have a promising route to proving universality of KPZ fluctuations (i.e. for general weight distributions).

Competition interface

For simplicity, assume ω_0 is continuous.

Consider the geodesics tree out of 0 and its subtrees out of e_1 and e_2 .

The two subtrees are separated by a **competition interface**.

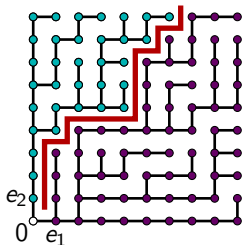


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Theorem. Assume g_{pp} differentiable at endpoints of its linear segments.

- The competition interface almost surely has a direction $\xi_*(\omega) \in \mathcal{U}$.
- ξ_* is supported everywhere outside the segments on which g_{pp} is linear.
- In this (random) direction $\xi_*(\omega)$, there are two geodesics out of 0, one through e_1 and one through e_2 .

(This generalizes part of Couplier's '11 results for exponential ω_0 to general weights. When ω_0 is exponential, there are no linear segments.)

Competition interface

Direction $\xi_*(\omega)$ has a nice characterization.

By monotonicity, $B^\xi(0, e_1) - B^\xi(0, e_2)$ decreases as $\xi \cdot e_1$ increases.

We have seen: $\forall \xi \in \mathcal{U}$ rational, $\mathbb{P}\{B^\xi(0, e_1) = B^\xi(0, e_2)\} = 0$.

Can show difference goes to ∞ as $\xi \rightarrow e_2$ and to $-\infty$ as $\xi \rightarrow e_1$.

Hence, $\exists! \xi_*(\omega)$ s.t.

$B^\xi(0, e_1) > B^\xi(0, e_2)$ (ξ -geodesics go to e_2) if $\xi \cdot e_1 < \xi_* \cdot e_1$ and

$B^\xi(0, e_1) < B^\xi(0, e_2)$ (ξ -geodesics go to e_1) if $\xi \cdot e_1 > \xi_* \cdot e_1$.

ξ_* is the direction of the competition interface.

Solvable case

When ω_0 is exponential (e.g. with rate 1) $B^\xi(0, e_1)$ and $B^\xi(0, e_2)$ are independent exponentials with known rates.

Then we can compute the distribution of ξ_* explicitly.

Switching to the angle θ_* that ξ_* makes with e_1 we recover Ferrari and Pimentel's result:

$$\mathbb{P}\{\theta_* \leq t\} = \frac{\sqrt{\sin t}}{\sqrt{\sin t} + \sqrt{\cos t}} \quad t \in [0, \pi/2].$$

Can also deal with the case when ω_0 is not continuous, and then get explicit results for ω_0 geometric.

Existence of Busemann functions

First, we get the existence of a process $\{B^\xi(x, y; \omega, \hat{\omega}) : \xi \in \mathcal{U}, x, y \in \mathbb{Z}^2\}$ on a larger (product) probability space.

This process is such that B^ξ is an L^1 stationary cocycle that recovers the original potential ω , and satisfies the monotonicity Busemann functions are supposed to satisfy.

This step is done by appealing to a fixed point result for a certain queuing operator from queuing theory.

Then, we couple gradients $G_{x, [n\xi]} - G_{y, [n\xi]}$ with that process, sandwiching them between B^ζ and B^η (plus a small error), where $\zeta \cdot e_1 < \xi \cdot e_1 < \eta \cdot e_1$.

This allows us to prove that the limiting Busemann function exists (at ξ as in standing assumption) and equals the B^ξ from queuing.

In particular, there was no need to extend the space.

Thank You

