Obliquely Reflected Brownian motions (ORBMs) in Non-Smooth Domains

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Why Study Obliquely Reflected Diffusions?

Applications

- limits of interacting particle systems such as TASEP;
- diffusion approximations of stochastic networks;
- rank-dependent diffusion models (in finance);
- biological models of gene networks;
- closely related to certain (non-reflecting) diffusions;

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Fundamental Mathematical Object

- a (non-symmetric) Markov process in a domain
- many basic questions are still not fully understood

Focus of this talk ...

Obliquely reflected Brownian motions (ORBMs)
in non-smooth (rough) domains
planar domains

Motivation

- Diffusions in fractal domains have unusual and interesting properties (Goldstein ('87), Kusuoka ('87), Barlow-Perkins ('88)
- loosely motivated by applications in biology

Challenge

Existing techniques for

- reflected diffusions in piecewise smooth domains
- normally reflected diffusions in fractal domains

are not applicable

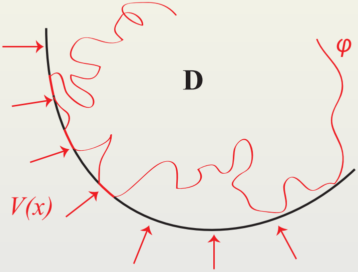
Acknowledgments

Draws on various joint works with

W. Kang
University of Maryland, Baltimore County
and
K. Burdzy, Z.-Q.-Chen and D. Marshall
University of Washington, Seattle

A Heuristic Description

Given a domain D with a vector field $v(\cdot)$ on the boundary ∂D, ORBM behaves infinitesimally like Brownian motion in the interior, is constrained to stay within the closure $\bar{D}$ of the domain and spends zero Lebesgue time on the boundary



Tools to study ORBMs in smooth domains

A. (Extended) Skorokhod problem (and stochastic differential equations with reflection – SDER)
B. Submartingale problem

A. The (Extended) Skorokhod Problem

The 1-dimensional Skorokhod Map $D = [0, \infty)$, $v(0) = e_1$

Definition (Skorokhod Problem (Skorokhod '61))

For every continuous $\mathbb{R}$-valued path ψ, find a continuous path ϕ s.t. $\forall t \geq 0$,

1. $\phi(t) \geq 0$ i.e., $\phi(t)$ lies in $[0, \infty)$
2. $\eta = \phi - \psi$ is non-decreasing
3. "η increases only when ϕ is on the boundary"

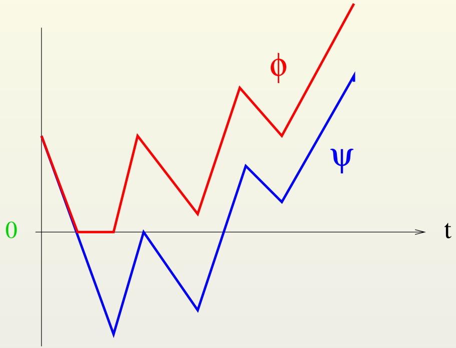
$$\int_0^\infty \phi(s) d\eta(s) = 0.$$

The Skorokhod Map Γ_0 on $[0, \infty)$

$\phi = \psi + \eta \geq 0$, η non-decreasing, $\int_0^\infty \phi(s) d\eta(s) = 0$.

An explicit formula (Skorokhod '61)

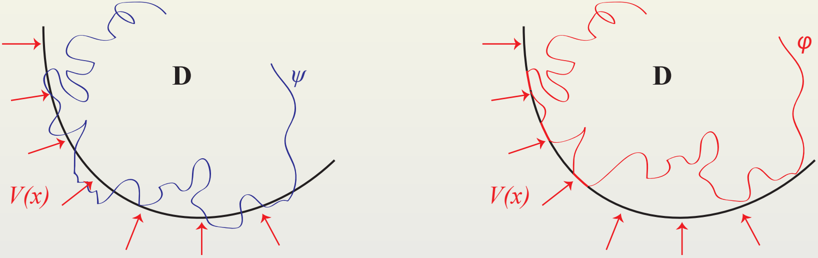
$$\phi = \psi + \eta \geq 0, \qquad \eta \text{ non-decreasing}, \qquad \int_0^\infty \phi(s) d\eta(s) = 0.$$



$$\phi(t) = \Gamma_0(\psi)(t) = \psi(t) + \sup_{s\in[0,t]} [-\psi(s)]^+$$

$Z = \Gamma_0(Z_0 + B)$ is *RBM* in 1-d

Obtain reflected Brownian motion as a constrained version of Brownian motion



Formulation of the Skorokhod Map in $\mathbb{R}^d$

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$$\int_0^\infty \phi(s) d\eta(s) = 0.$$

1. Property 2 is equivalent to $\eta(t) - \eta(s) \geq 0$ for all $0 \leq s \leq t$;
2. Setting $v(x) = 0$ if $x > 0$, properties 2 and 3 are equivalent to

$$\eta(t) - \eta(s) \in \overline{\text{co}} \left(\cup_{u \in (s,t]} v(\phi(u)) \right),$$

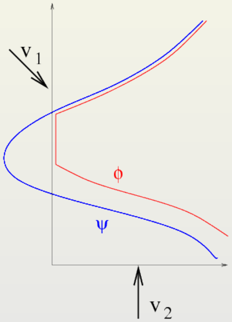
where, for $A \subset \mathbb{R}^d$, $\overline{\text{co}}[A]$ is the closure of the convex cone generated by the vectors in A.

The Multidimensional Skorokhod Problem

Obtain reflected Brownian motion as a constrained version of Brownian motion

Natural to consider piecewise smooth domains where v is multi-valued

$$v(0) = \{\alpha_1 v_1 + \alpha_2 v_2 : \alpha_1, \alpha_2 \geq 0\}$$



The Extended Skorokhod Map on $(D, v(\cdot))$

Extend v to $\bar{D}$ by setting $v(x) = 0$ for $x \in D$

Definition (Extended Skorokhod Problem ('R '06))

For every continuous $\mathbb{R}^d$-valued path ψ, find a continuous ϕ s.t. $\forall t \geq 0$,

1. $\phi(t) \in \overline{D}$;
2. $\eta = \phi - \psi$ satisfies for every $0 \leq s \leq t$,

$$\eta(t) - \eta(s) \in \overline{\text{co}} \left(\cup_{u \in (s,t]} v(\phi(u)) \right),$$

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Previous Formulations and Results

Tanaka ('79), Harrison-Reiman ('81), Lions-Sznitman ('84), Bernard El-Kharroubi ('91), Dupuis-Ishii ('91), Costantini ('92), Dupuis-Ramanan ('99), ...

The Extended Skorokhod Map on $(D, v(\cdot))$

- Previous formulations of the Skorokhod Map assumed η is of bounded variation;
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- The ESP formulation enabled one to construct solutions to SDER that are not necessarily semimartingales, thus extending the applicability of the SDER approach.
- The ESP can be used to construct both strong and weak solutions to the associated SDER.
- Construction of semimartingale reflected Brownian motions using SDER: Costantini, Reiman-Williams (1988), Taylor-Williams (1993) for polyhedral domains

B. The submartingale problem

The Submartingale Problem (Stroock-Varadhan '71)

Given $(D, v(\cdot))$, b, σ, find probability measures $\mathbb{Q}_z$, $z \in \bar{D}$, on $\mathcal{C}([0,\infty) : \mathbb{R}^n)$ such that

- For every $z \in \bar{D}$, $\mathbb{Q}_z(w(0) = z) = 1$;
- Under each $\mathbb{Q}_z$,

$$M_t^f \doteq f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\, ds$$

is a submartingale for all $f \in \mathcal{H}_0$, where

$$\mathcal{H}_0 = \{f \in \mathcal{C}_b^2(D) : \langle Df(x), v(x)\rangle \geq 0\}$$

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Well-posedness of the submartingale problem

The submartingale problem is said to be well posed if there exists a solution to the submartingale problem and it is unique.

Results on the Submartingale Problem

1. Stroock and Varadhan (1971) established well-posedness of the submartingale problem for bounded $\mathcal{C}^2$ domains with Lipschitz continuous reflection field v satisfying $|\nabla v| \geq 1$.
2. Extended to specific non-smooth domains in Varadhan-Williams (1985); Williams (1987); Deblassie (1987, 1996); Deblassie-Toby (1993), ...

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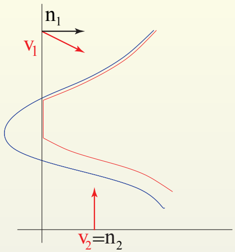
2. Extended to specific non-smooth domains in Varadhan-Williams (1985); Williams (1987); Deblassie (1987, 1996); Deblassie-Toby (1993), ...
3. A general multi-dimensional formulation was not available ... cited as an open problem (Williams 1995, DeBlassie 1997)

Characteristics of Non-smooth Domains



• The direction vector field $v(\cdot)$ can be multi-valued.

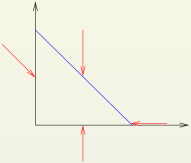
• For piecewise smooth domains $D = \cap_i D_i$, with each domain having a smooth vector field v_i, v on the intersections of multiple smooth boundaries is defined as

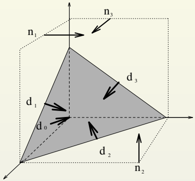
$$v(x) = \left\{ \sum_{i \in I(x)} \alpha_i v_i(x), \alpha_i \geq 0 \right\},$$

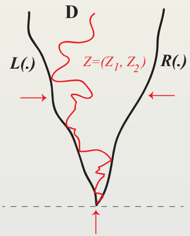
Challenging Aspects of Non-smooth Domains

A set $\mathcal{V}$ of irregular points where v contains a half-plane

$$\mathcal{V} = \partial D \setminus \{x \in \partial D : \exists n \in n(x) \text{ such that } \langle n, v \rangle > 0, \forall v \in v(x)\}$$







The Submartingale Problem for Non-smooth Domains

Submartingale Formulation (Kang-'R, '12)

Given $(D, v(\cdot))$, b, σ, find probability measures $\mathbb{Q}_z$, $z \in \bar{D}$, on $\mathcal{C}([0,\infty) : \mathbb{R}^n)$ such that

- $\mathbb{Q}_z(\omega(0) = 0) = 1$
-
$$M_t^f \doteq f(X_t) - \int_0^t \mathcal{L}f(X_s)\, ds, \quad t \geq 0,$$

  is a $\mathbb{Q}_z$-submartingale for all $f \in \mathcal{H}$:

$$\mathcal{H} \doteq \left\{ f \in \mathbb{C}_c^2(\bar{D}) \oplus \mathbb{R} : \quad \begin{array}{l} f \text{ is constant in a neighborhood of } \mathcal{V}, \\ \langle v, \nabla f(y) \rangle \geq 0 \text{ for } v \in v(y) \text{ and } y \in \partial D \end{array} \right\}$$

- For every $z \in \bar{D}$, $\mathbb{Q}_z$-almost surely,

$$\mathcal{L}eb\{s \in [0,\infty) : \omega(s) \in \mathcal{V}\} = 0.$$

An Equivalence Theorem

- Stroock and Varadhan (1969) introduced the martingale problem for diffusions and showed, under general conditions on b and σ that it was equivalent to the SDE formulation
- Reflected diffusions can also be defined as solutions to SDERs using the extended Skorokhod map Γ
- Is there a similar equivalence between SDERs and the submartingale formulation here?

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Theorem (Kang-'R '13)

When the set $\mathcal{V}$ is finite, D is piecewise $\mathcal{C}^2$ and v is $\mathcal{C}^1$, b bounded and measurable and σ continuous, then well-posedness of submartingale formulation is equivalent to existence and uniqueness in law of weak solutions to SDERs

Comments on the Proof

- Main thrust: need to construct a weak solution from a solution to the submartingale problem
- For the martingale formulation, use test functions $f(x) = x_i$, $f(x) = x_i x_j$ to show that

$$f(X_t) - f(X_0) - \int_0^t \langle \nabla f(X_s), b(X_s) \rangle \, ds$$

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  is a martingale, then use the martingale representation theorem
- Here, choice of test functions limited by derivative conditions
  - Construction depends on geometry; especially at intersections of faces;
  - need to identify the "local time" part;
  - study behavior of quadratic variation of the mgale part of the Doob decomposition on the boundary;

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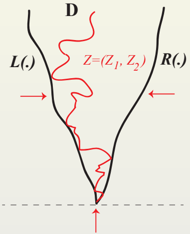
- Proof much more subtle ...
- In fact, the equivalence fails if $\mathcal{V}$ is not a finite set

$$D = \{y \in \mathbb{R}^3 : y_2 \geq 0, L(y) \leq y_2 \leq R(y)\}$$



- Suggests that additional conditions may need to be imposed near $\mathcal{V}$ ...
- Are there a natural set of conditions ?

Other Questions Related to Reflecting Diffusions

- semimartingale property
- characterization of stationary Distributions
- hitting edges and corners
- ...

RBMs in Non-smooth (Fractal) Domains

How can one even define normally reflected BMs in fractal domains?

Challenge

No way to make sense of the normal vector field

Normally Reflected ORBMs

Dirichlet form approach

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- A pair $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, where
  - $\mathcal{D}(\mathcal{E})$ is a dense linear subspace of $\mathbb{L}^2(\mathcal{E}; m)$;
  - $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \mapsto \mathbb{R}$ is a bilinear form;

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  - $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \mapsto \mathbb{R}$ is a bilinear form;
- that is symmetric: $\mathcal{E}(u, v) = \mathcal{E}(v, u), \forall u, v \in \mathcal{D}(\mathcal{E})$;
- and closed, i.e.,
  - $\mathcal{E}(u, u) \geq 0$;
  - $\mathcal{D}(\mathcal{E})$ is a Hilbert space when equipped with the norm $\mathcal{E}(u, v) + (u, v)_{\mathbb{L}^2}$
- satisfies the contraction property

$$\mathcal{E}(u_*, u_*) \leq \mathcal{E}(u, u), \quad u_* = \min(\max(u, 0), 1).$$

Dirichlet form approach

The "energy" functional $\mathcal{E}$ is used to define a Markov process $X_t, t \geq 0$:

$$\mathcal{E}(u,u) = \lim_{t \downarrow 0} \frac{1}{2t} \int_E \mathbb{E}_z \left[(u(X_t) - u(X_0))^2 \right] m(dz).$$

In the other direction,

$$(\mathcal{E}, \mathcal{D}(\mathcal{E})) \mapsto (T_t) \mapsto (p_t) \mapsto \{X_t\}_{t \geq 0}$$

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Results using the Dirichlet Form Approach

- Beurling and Deny (1959)
- Silverstein and Fukushima (1970s)
- Fukushima (1990s): If a Dirichlet form on a locally compact state space is regular, one can construct an associated Markov process with RCLL paths.

Normal RBMs in Fractal Domains

- Normal RBMs are symmetric Markov processes;
- Dirichlet form techniques have been successively used to construct normally reflected Brownian motions in quite general domains (Z.-Q.-Chen '93)

ORBMs in Fractal Domains

Can the Dirichlet form approach be applied to ORBMs?

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- Ma and Röckner ('92): a more general result relating (non-symmetric) Dirichlet forms with Markov processes;

ORBMs in Fractal Domains

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- ORBMs are not symmetric Markov processes;
- There exist extensions to the Dirichlet form approach that relax the symmetry assumption (sector condition);
- Ma and Röckner ('92): a more general result relating (non-symmetric) Dirichlet forms with Markov processes;
- However, not much success with ORBMs.

Challenges

- The normal and tangential vector fields are not well defined in the classical sense
- ORBM is not a symmetric Markov process
- A new approach is required ...

ORBMs in Smooth Planar domains

- Parametrize ORBMs in smooth domains by "angle of reflection"
- Let B be standard two-dimensional Brownian motion
- Given $D \subset \mathbb{C}$ a smooth bounded open set and $\theta : \partial D \mapsto (-\pi/2, \pi/2)$ Borel measurable function satisfying $\sup_{x \in \partial D} |\theta(x)| < \pi/2$.
- $\mathbf{n}(x)$: unit inward normal vector at $x \in \partial D$
- $\mathbf{t}(x)$: unit tangent vector to ∂D at x
- Vector field v_θ on ∂D associated with θ:

$$\mathbf{v}_\theta(x) = \mathbf{n}(x) + \tan\theta(x)\mathbf{t}(x)$$

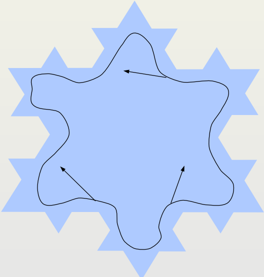
- Parametrize vector field in terms of the angle of reflection θ

A. Domain Approximation

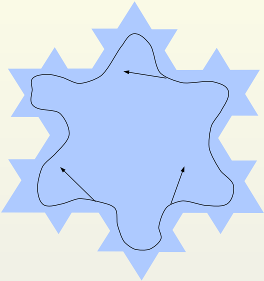
- Given a simply connected Jordan domain D, $y_0 \in D$, approximate it by a sequence of smooth domains D^k in the sense that for all k,

$$y_0 \in D_k \subset D_{k+1} \subset D, \text{ and } \bigcup_k D_k = D$$

- For each k consider a smooth vector field θ^k and let Z^k be ORBM associated with (D^k, θ^k).



ORBMs in Planar Domains: A. Domain Approximation



- When does Z^k converge to some limit process Z, and in what sense ?
- Is the “limit” Z an ORBM in D in any reasonable sense?
- Is there an independent characterization of the limit ORBM ?

Possible Constructions: B. Conformal Mappings

- Let D_* denote the unit disc in the plane

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- Let D_* denote the unit disc in the plane
- First define the ORBM on the unit planar disc D_*
- Then use conformal maps to extend the definition to more general domains

Possible Constructions: B. Conformal Mappings

ORBMs in the unit planar disc D_*

- Recall, given $\theta : \partial D \mapsto (-\pi/2, \pi/2)$,

$$\mathbf{v}_\theta(x) = \mathbf{n}(x) + \tan\theta(x)\mathbf{t}(x)$$

- When θ is $\mathcal{C}^2$, D smooth, Skorokhod Map Γ is well defined; RBM

$$Z = \Gamma(Z_0 + B)$$

or, equivalently,

$$Z_t = Z_0 + B_t + \int_0^t \mathbf{v}_\theta(Z_s)dL_s,$$

Here, L is the local time of X on ∂D.

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- When θ is $\mathcal{C}^2$, D smooth, Skorokhod Map Γ is well defined; RBM

$$Z = \Gamma(Z_0 + B)$$

or, equivalently,

$$Z_t = Z_0 + B_t + \int_0^t \mathbf{v}_\theta(Z_s) dL_s,$$

Here, L is the local time of X on ∂D.

- When $D = D_*$, strong solution exists without regularity assumption on θ (use polar decomposition)

ORBMs in Planar Domains: Possible Constructions

B. Conformal Mapping

$$\mathcal{T} = \{\theta \in L^\infty(\partial D_*) : ||\theta||_\infty \leq \pi/2,\ \theta \not\equiv \pi/2,\ \text{and}\ \ \theta \not\equiv -\pi/2\}.$$

Suppose $\theta \in \mathcal{T}$ is $\mathcal{C}^2$ and let Z be a $(D_*, \mathbf{v}_\theta)$ ORBM.

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- Let D be a simply connected bounded domain
- Let $f : D_* \mapsto D$ be a one-to-one onto analytical function.

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- Let D be a simply connected bounded domain
- Let $f : D_* \mapsto D$ be a one-to-one onto analytical function.
- Define

$$c(t) = \int_0^t |f'(Z_s)|^2 ds, \qquad \text{for } t \geq 0,$$

$$\zeta = \inf\{t \geq 0 : c(t) = \infty\},$$

$$Y(t) = f(Z_{c^{-1}(t)}), \qquad \text{for } t \in [0, \zeta).$$

B. Conformal Mapping (contd.)

- Z be a $(D_*, \mathbf{v}_\theta)$ ORBM.
- Let $f : D_* \mapsto D$ be a one-to-one onto analytical function.
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- Is Y an ORBM in a suitable sense?

Results: A. Smooth domain approximation

$D \subset \mathbb{R}^2$ – open bounded simply connected set
$D_k \subset D_{k+1}$, $\bigcup_k D_k = D$, D_k have smooth boundaries
$\theta_k(x)$ – reflection angle; $x \in \partial D_k$
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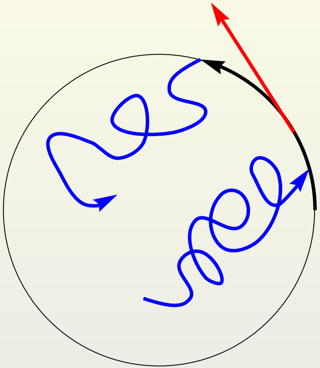
How does one independently characterize the ORBM?

Jumps on the boundary when $\theta(x) = \pi/2$

- Limit process could have jumps



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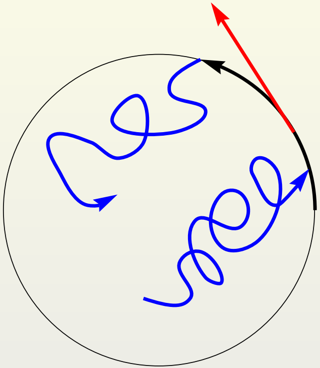

- So convergence of Z^k to Z is (in general) in a certain M_1 topology
- Limit could be "excursion reflected Brownian motion (ERBM)" when limit $\theta \equiv \pi/2$

Towards an independent characterization

Alternative parametrization of ORBMs in D_*

D_* – unit disc in $\mathbb{R}^2$
$\theta(x)$ – angle of reflection at $x \in \partial D$

$$\mathcal{T} = \{\theta \in L^\infty(\partial D_*) : ||\theta||_\infty \leq \pi/2,\ \theta \not\equiv \pi/2,\ \text{and}\ \ \theta \not\equiv -\pi/2\}.$$

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$$\theta \in \mathcal{T} \leftrightarrow (h, \mu_0) \in \mathcal{H} \times \mathbb{R}$$

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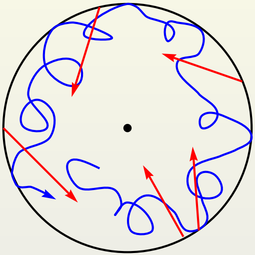
and

Rate of Rotation μ_0

D_* – unit disc in $\mathbb{R}^2$, $\theta(x)$ – angle of reflection at $x \in \partial D$

$$\theta \leftrightarrow (h, \mu_0)$$

$h(x)dx$ – stationary distribution $\qquad$ μ_0 – rate of rotation around zero



$$\lim_{t\to\infty} \frac{1}{t} \arg X_t - \mu_0 \Rightarrow \text{Cauchy}.$$

$$\lim_{t\to\infty} \frac{1}{t} \arg^* X_t = \mu_0$$

Alternative Parametrization of ORBMs

$$\theta \leftrightarrow (h, \mu_0)$$

The correspondence has quite an explicit form

THEOREM (forthcoming; Burdzy, Chen, Marshall, 'R)

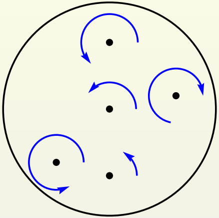
$$h(z) = \frac{\mathrm{Re}\, \exp(\widetilde{\theta}(z) - i\theta(z))}{\pi \mathrm{Re}\,(e^{-i\theta(0)})} = \frac{\mathrm{Re}\, \exp(\widetilde{\theta}(z) - i\theta(z))}{\pi \cos\theta(0)}, \qquad z \in D,$$

$$\mu_0 = \tan\theta(0) = \int_D \tan\theta(z) h(z) dz,$$

$$\theta(z) = -\arg\left(h(z) + i\widetilde{h}(z) - i\mu/\pi\right), \qquad z \in D.$$

Rate of Rotation vector field $z \mapsto \mu(z)$

- We can also parameterize the ORBM in terms of “rotation rates”



- On D_*, this correspondence takes the following form

$$\theta \in \mathcal{T} \leftrightarrow \mu(\cdot) \in \mathcal{R},$$

$$\mathcal{R} = \{\mu : \mu \text{ is harmonic in } D_* \text{ and } \tilde{\mu}(z) > -1, \quad \text{for all } z \in D_*\}.$$

- Again, $\mu(z)$ can be written quite explicitly in terms of θ.
- Probabilistic interpretation of $\mu(z)$:

$$\lim_{t \to \infty} \arg^* (X_t - z)/t = \mu(z).$$

ORBMs in Simply Connected Planar Domains

Theorem (forthcoming; Burdzy, Chen, Marshall, 'R)

- *For any simply connected bounded domain D and* $\theta \in \mathcal{T}$, *we can define an ORBM Y using a conformal mapping f as described earlier.*
- *If* $\theta \leftrightarrow (h, \mu)$, $\theta \leftrightarrow \mu$, *then Y has stationary density*

$$\widehat{h} = h \circ f^{-1} / ||h \circ f^{-1}||_1^D,$$

and

$$\lim_{t \to \infty} \arg^* \frac{Y_t - z}{t} = \frac{\mu(f^{-1}(z))}{||h \circ f^{-1}||_1^D}. \tag{1}$$

- *For any* $\mu_0 \in \mathbb{R}$ *and* $\widehat{h}$ *a positive harmonic function in D with* $||\widehat{h}||_1 = 1$, *there exists an ORBM Y in D such that Y has stationary distribution* $\widehat{h}$ *and* (1) *holds with* $\mu(\cdot) \leftrightarrow (\mu_0, h)$.

Consistent with Approximations

THEOREM (forthcoming; Burdzy, Chen, Marshall, 'R)

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[

Recall: forthcoming; Burdzy, Chen, Marshall, 'R] The limit process Z can be characterized in terms of θ and D as described above. And this characterization is consistent with the ORBM obtained in terms of the conformal mapping.

Summary

- Reflected Diffusions in piecewise-smooth domains arise in a variety of fields ranging from math physics and finance to queueing theory.
- A new paradigm has been developed for characterization of ORBMs in bounded planar domains, including some ORBMs with jumps (excursion-reflected Brownian motions)
- Many questions remain regarding the construction of ORBMs in more general (multiply connected) planar domains as well as multi-dimensional domains
- Several foundational questions remain even for RBMs in polyhedral domains

List of Some of My Relevant Works

- K. Burdzy, Z.-Q. Chen, D. Marshall and K. Ramanan, "Obliquely reflected Brownian motions in non-smooth planar domains," forthcoming, 2014.
- W. Kang and K. Ramanan, "On the Submartingale Problem for reflected diffusions in piecewise smooth domains", Preprint, 2014.
- W. Kang and K. Ramanan, "Characterizations of stationary distributions of reflected diffusions," Ann. Appl. Probab., 2014.
- W.N. Kang and K. Ramanan, "A Dirichlet process characterization of a class of reflected diffusions," Ann. Probab., **38** (2010) 1062–1105.
- K. Burdzy, W.N. Kang and K. Ramanan, "The Skorokhod map in a time-dependent interval," Stoch. Proc. Appl., **119** (2009) 428–452.
- K. Ramanan. "Reflected diffusions defined via the extended Skorokhod map." Elec. Jour. Probab., **11** (2006), 934–992.