LAN property for ergodic jump-diffusion processes with discrete observations

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Parametric statistical model

- Consider a real-valued Markov process $X^\theta = (X_t^\theta, t \geq 0)$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and adapted to the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of a standard Brownian motion $B = (B_t, t \geq 0)$.
- Let P^θ be the probability measure induced by X^θ.
- The unkown parameter θ belongs to Θ a closed rectangle of $\mathbb{R}^k$.
- Consider a high frequency observation of X^θ denoted $X^n = (X_{t_0}, X_{t_1}, ..., X_{t_n})$, where $t_k = k\Delta_n$:
  - Distance between observations $\Delta_n \to 0$, as $n \to \infty$.
  - Horizon $n\Delta_n \to \infty$, as $n \to \infty$.
- Let P_n^θ the probability measure on induced by X^n under θ.
- Assume that X^n admits a density $p_n(x; \theta)$.
- Let $p^\theta(t - s, x, y)$ the transition density of X_t^θ at y conditionally on $X_s^\theta = x$.

The LAN property

The parametric statistical model $\{\mathrm{P}_n^{\theta}, \theta \in \Theta\}$ is said to have the local asymptotic normality (LAN) property at θ if there exist

1. a k-dimensional vector $\varphi_n(\theta)$ with strictly positive entries tending to zero as $n \to \infty$
2. a $k \times k$ symmetric positive definite matrix $\Gamma(\theta)$

such that for any $u \in \mathbb{R}^k$, as $n \to \infty$,

$$\log \frac{d\mathrm{P}_n^{\theta+\varphi_n(\theta)u}}{d\mathrm{P}_n^{\theta}}(X^n) \xrightarrow{\mathcal{L}(\mathrm{P}^{\theta})} u^{\mathsf{T}}\mathcal{N}(0,\Gamma(\theta)) - \frac{1}{2}u^{\mathsf{T}}\Gamma(\theta)u.$$

$\mathcal{N}(0,\Gamma(\theta))$: centered $\mathbb{R}^k$-valued Gaussian variable with covariance $\Gamma(\theta)$.
$\Gamma(\theta)$:asymptotic Fisher information matrix.

References : Le Cam'60, Hájek'70.

Extension to $\Gamma(\theta)$ random (mixed normal, LAMN) : Jeganathan'82-83

Remarks

- Observe that the LAN property is equivalent to

$$\log \frac{d\mathrm{P}_n^{\theta+\varphi_n(\theta)u}}{d\mathrm{P}_n^{\theta}}(X^n) = \log \frac{p_n(X^n;\theta+\varphi_n(\theta)u)}{p_n(X^n;\theta)}$$
$$= u^{\mathsf{T}}\varphi_n(\theta)\nabla_\theta \log p_n(X^n;\theta) - \frac{1}{2}u^{\mathsf{T}}\Gamma(\theta)u + o_{\mathrm{P}^\theta}(1),$$

  where $\varphi_n(\theta)\nabla_\theta \log p_n(X^n;\theta)$ converges in P^θ-law to $\mathcal{N}(0,\Gamma(\theta))$.

- Assume $k=1$. By the Markov property

$$\log \frac{p_n(X^n;\theta+\varphi_n(\theta)u)}{p_n(X^n;\theta)} = \sum_{k=0}^{n-1}\varphi_n(\theta)u\int_0^1 \partial_\theta \log p^{\theta(\ell)}(\Delta_n, X_{t_k}, X_{t_{k+1}})d\ell$$

  where $\theta(\ell) := \theta + \ell\varphi_n(\theta)u$.

Expression of the log-likelihood via Malliavin calculus

Theorem (Malliavin, Gobet'01)

Suppose that $X_t^\theta \in \mathbb{D}^{1,2}$ for all $t \geq 0$, and there exists a stochastic process $u \in$Dom(δ) such that

$$\int_{t_i}^{t_{i+1}} D_t X_{t_i}^\theta u(t) dt = \partial_\theta X_{t_i}^\theta, \qquad \forall\, i = 0, ..., n-1,$$

Then, P^θ-a.s.,

$$\partial_\theta \log p^\theta(\Delta_n, X_{t_k}, X_{t_{k+1}}) = \mathrm{E}_{X_{t_k}}^\theta \left[\delta(u) \middle| X_{t_{k+1}}^\theta = X_{t_{k+1}} \right].$$

Proof If φ is a test function, using the chain rule and the duality relation,

$$\begin{aligned} \mathrm{E}_{X_{t_k}}^\theta \left[\partial_\theta \varphi(X_{t_{k+1}}^\theta) \right] &= \mathrm{E}_{X_{t_k}}^\theta \left[\varphi'(X_{t_{k+1}}^\theta) \cdot \partial_\theta X_{t_{k+1}}^\theta \right] = \mathrm{E}_{X_{t_k}}^\theta \left(\langle D(\varphi(X_{t_{k+1}}^\theta)), u \rangle_2 \right) \\ &= \mathrm{E}_{X_{t_k}}^\theta \left(\varphi(X_{t_{k+1}}^\theta) \delta(u) \right). \end{aligned}$$

On the other hand,

$$\mathrm{E}_{X_{t_k}}^\theta \left[\partial_\theta \varphi(X_{t_{k+1}}^\theta) \right] = \mathrm{E}_{X_{t_k}}^\theta \left(\varphi(X_{t_{k+1}}^\theta) \partial_\theta \log p^\theta(\Delta_n, X_{t_k}, X_{t_{k+1}}^\theta) \right).$$

Consequences of the LAN property

Assume that $\{\mathrm{P}_n^{\theta}, \theta \in \Theta\}$ satisfies the LAN property.

- A sequence of estimators $(\theta_n)_{n\geq 1}$ is called regular if for all $u \in \mathbb{R}^k$,

$$\varphi_n(\theta)^{-1}\left(\theta_n - (\theta + \varphi_n(\theta)u)\right) \overset{\mathcal{L}(\mathrm{P}^{\theta+\varphi_n(\theta)u})}{\longrightarrow} V(\theta), \quad \text{as } n \to \infty,$$

  for some $\mathbb{R}^k$-valued random variable $V(\theta)$.

- Conditional convolution theorem : Let $(\theta_n)_{n\geq 1}$ a sequence of regular estimators of θ. Then

$$\mathcal{L}\left(V(\theta)\right) = \mathcal{N}\left(0, \Gamma(\theta)^{-1}\right) \star G_{\Gamma(\theta)},$$

  where $G_{\Gamma(\theta)}$ is independent of $\mathcal{N}$.

- A sequence of estimators $(\theta_n)_{n\geq 1}$ is called asymptotically efficient if

$$\varphi_n^{-1}(\theta)\left(\theta_n - \theta\right) \overset{\mathcal{L}(\mathrm{P}^{\theta})}{\longrightarrow} \mathcal{N}\left(0, \Gamma(\theta)^{-1}\right), \quad \text{as } n \to \infty.$$

- Minimax theorem : $\Gamma(\theta)^{-1}$ is a lower bound for the asymptotic covariance of any sequence of unbaised estimators.

LAN property for diffusion processes

1. Gobet'01 derives the LAMN property in the non-ergodic case :

$$X_t^{\theta} = X_0 + \int_0^t b(\theta, s, X_s^{\theta})ds + \int_0^t \sigma(\theta, s, X_s^{\theta})dB_s, \quad t \in [0, 1].$$

2. Gobet'02 shows the LAN property in the ergodic case :

$$X_t^{\theta,\beta} = X_0 + \int_0^t b(\theta, X_s^{\theta,\beta})ds + \int_0^t \sigma(\beta, X_s^{\theta,\beta})dB_s, \quad t \geq 0.$$

3. Delattre and *al.*'11 have established the LAMN property :

$$X_t^{\lambda} = X_0 + \int_0^t b(s, X_s^{\lambda})ds + \int_0^t a(s, X_s^{\lambda})dB_s + \sum_{k:T_k \leq t} c(X_{T_k-}^{\lambda}, \lambda_k),$$

   for $t \in [0, 1]$, where the jump times $T_1, T_2, \ldots, T_K$ are given.

4. Kawai'13, LAN for Ornstein-Uhlenbeck Processes with Jumps

The density estimates

- These papers use upper and lower Gaussian type estimates of the transition densities of the diffusion processes.
- Some computations show that in simpler one dimensional situations (Gaussian type jumps) the upper density estimate is of the type

$$\frac{C}{\sqrt{t}} \exp\left(-c|y-x|\sqrt{\left|\ln \frac{|y-x|}{t}\right|}\right),$$

and the lower density estimates are of the type

$$Ce^{-\lambda t} \exp\left(-c|y-x|\sqrt{\left|\ln \frac{|y-x|}{t}\right|}\right),$$

with a different estimate over the diagonal.
- This shows that the upper and lower bounds are of different characteristic and Gobet's argument cannot be implemented.

$$X_t = x + \theta t + \sigma B_t + N_t - \lambda t.$$

Theorem

For all $z = (u, v, w) \in \mathbb{R}^3$*, as* $n \to \infty$*,*

$$\log \frac{p(X^n; (\theta_n, \sigma_n, \lambda_n))}{p(X^n; (\theta, \sigma, \lambda))} \xrightarrow{\mathcal{L}(\mathrm{P}_x^{\theta,\sigma,\lambda})} z^{\mathsf{T}} \mathcal{N}(0, \Gamma(\theta, \sigma, \lambda)) - \frac{1}{2} z^{\mathsf{T}} \Gamma(\theta, \sigma, \lambda) z,$$

where $\mathcal{N}(0, \Gamma(\theta, \sigma, \lambda))$ *is a centered* $\mathbb{R}^3$*-valued Gaussian vector with covariance matrix*

$$\Gamma(\theta, \sigma, \lambda) = \frac{1}{\sigma^2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 + \frac{\sigma^2}{\lambda} \end{pmatrix},$$

and we set $\theta_n := \theta + \frac{u}{\sqrt{n\Delta_n}}, \sigma_n := \sigma + \frac{v}{\sqrt{n}}, \lambda_n := \lambda + \frac{w}{\sqrt{n\Delta_n}}$.

Sketch of the proof

1. Markov's property, the integration by parts formula of the Malliavin calculus and Girsanov's theorem, allows to obtain en expansion of the log-likelihood of the form

$$\log \frac{p\left(X^n; (\theta_n, \sigma_n, \lambda_n)\right)}{p\left(X^n; (\theta, \sigma, \lambda)\right)} = \sum_{k=0}^{n-1} \left(\xi_{k,n} + H_{k,n} + \eta_{k,n} + M_{k,n} + \beta_{k,n} - R_{k,n}\right).$$

2. Apply a central limit theorem for triangular arrays of random variables to $\xi_{k,n} + \eta_{k,n} + \beta_{k,n}$ to show the convergence in law.

3. Negligible terms : the key argument consists in conditioning on the number of jumps $J_j := \{N_{t_{k+1}} - N_{t_k} = j\}$ within the conditional expectation which expresses the transition density and outside it.

4. Show a large deviation type inequality.

Sketch of the proof

Large deviation principle : Set

$$M_{1,p}^{\bar{\theta},\bar{\sigma},\bar{\lambda}} := \sum_{j=0}^{\infty} j^p \mathrm{E}_{X_{t_k}}^{\theta,\sigma,\lambda}\left[\mathbf{1}_{J_j}\mathrm{E}_{X_{t_k}}^{\bar{\theta},\bar{\sigma},\bar{\lambda}}\left[\mathbf{1}_{J_j^c}\middle| X_{t_{k+1}}^{\bar{\theta},\bar{\sigma},\bar{\lambda}} = X_{t_{k+1}}\right]\right],$$

$$M_{2,p}^{\bar{\theta},\bar{\sigma},\bar{\lambda}} := \sum_{j=0}^{\infty} \mathrm{E}_{X_{t_k}}^{\theta,\sigma,\lambda}\left[\mathbf{1}_{J_j}\mathrm{E}_{X_{t_k}}^{\bar{\theta},\bar{\sigma},\bar{\lambda}}\left[\mathbf{1}_{J_j^c}\left(N_{t_{k+1}} - N_{t_k}\right)^p \middle| X_{t_{k+1}}^{\bar{\theta},\bar{\sigma},\bar{\lambda}} = X_{t_{k+1}}\right]\right].$$

Lemma Assume that $|\theta - \bar{\theta}| \leq \frac{C}{\sqrt{n\Delta_n}}$ and $|\lambda - \bar{\lambda}| \leq \frac{C}{\sqrt{n\Delta_n}}$. Then for all $\alpha \in (0, \frac{1}{2})$, and n large enough,

$$M_{1,p}^{\bar{\theta},\bar{\sigma},\bar{\lambda}} + M_{2,p}^{\bar{\theta},\bar{\sigma},\bar{\lambda}} \leq C_1 e^{-\frac{1}{C_2\Delta_n^{1-2\alpha}}}.$$

Idea of the proof : Separate the cases $\mathbf{1}_{\{|B_{t_{k+1}} - B_{t_k}| \leq \Delta_n^\alpha\}}$ and $\mathbf{1}_{\{|B_{t_{k+1}} - B_{t_k}| > \Delta_n^\alpha\}}$.

An ergodic diffusion process with jumps

$$dX_t^\theta = b(\theta, X_t^\theta)dt + \sigma(X_t^\theta)dB_t + \int_{\mathbb{R}_0} c(X_{t-}^\theta, z)\,(N(dt, dz) - \nu(dz)dt)$$

- Usual Lipschitz, ellipticity, ergodicity and regularity assumptions.
- $N(dt, dz)$ is a Poisson random measure with intensity measure $\nu(dz)dt$, where $\lambda = \int_{\mathbb{R}_0} \nu(dz) < \infty$ independent of B such that
- There exist constants $q > 1$, $\rho_1, \rho_2 > 0$ and $0 < \upsilon, \gamma < \frac{1}{2}$ such that as $n \to \infty$,

$$\sqrt{n}\left(n\Delta_n\left(\int_{\{|z|\geq \rho_2 \Delta_n^{-\gamma}\}} \nu(dz) + \int_{\{|z|\leq \rho_1 \Delta_n^{\upsilon}\}} \nu(dz)\right)\right)^{\frac{1}{q}} \to 0.$$

- There exists $n_0 \geq 1$ such that for all $\theta \in \Theta$, there exists $C > 0$ such that

$$\sup_{n\geq n_0} \max_{k\in\{0,\dots,n\}} \mathrm{E}_x^\theta\left[e^{C\Delta_n^{1-2\gamma} X_{t_k}^2}\right] < \infty.$$

Theorem

For all $\theta \in \Theta$ and $u \in \mathbb{R}$, as $n \to \infty$,

$$\log \frac{p(X^n; \theta_n)}{p(X^n; \theta)} \xrightarrow{\mathcal{L}(\mathrm{P}_X^\theta)} u\mathcal{N}\left(0, \Gamma(\theta)\right) - \frac{u^2}{2}\Gamma\left(\theta\right),$$

where $\theta_n = \theta + \frac{u}{\sqrt{n\Delta_n}}$, and $\mathcal{N}(0, \Gamma(\theta))$ is a centered Gaussian random variable with variance

$$\Gamma\left(\theta\right) = \int_{\mathbb{R}} \left(\frac{\partial_\theta b(\theta, x)}{\sigma(x)}\right)^2 \pi_\theta(dx).$$

Another diffusion process with jumps

$$dX_t^{\theta,\beta} = b(\theta, X_t^{\theta,\beta})dt + \sigma(\beta, X_t^{\theta,\beta})dB_t + dZ_t - \lambda \int_{\mathbb{R}_0} z\mu(dz)dt$$

- Z_t is a compound Poisson process independent of B rate $\lambda > 0$ and jump size distribution $\mu(dz) = \sum_{i=1}^{\infty} p_{a_i}\delta_{a_i}(dz)$, $a_i \in \mathbb{R}_0$, $0 \leq p_{a_i} \leq 1$, $\sum_{i=1}^{\infty} p_{a_i} = 1$, and for any $p > 1$, $\sum_{a_i} p_{a_i}^{\frac{1}{p}} < \infty$.
- For any $\omega, \omega' \in \Omega$

$$\left|\left(Z_{t_{k+1}} - Z_{t_k}\right)(\omega) - \left(Z_{t_{k+1}} - Z_{t_k}\right)(\omega')\right| \begin{cases} = 0, & \text{or} \\ \geq C\Delta_n^{\upsilon}. \end{cases}$$

Furthermore, for all $q > 1$ and $p \in \{2, 4\}$,

$$\sum_r r^p \mathrm{P}_{X_{t_k}}^{\theta,\beta}\left(Z_{t_{k+1}} - Z_{t_k} = r\right)^{\frac{1}{q}} < \infty, \sum_r \mathrm{P}_{X_{t_k}}^{\theta,\beta}\left(Z_{t_{k+1}} - Z_{t_k} = r\right)^{\frac{1}{q}} < \infty.$$

Example : Assume that the jump sizes are positive and for some constant $c > 0$, $\int_0^{\infty} e^{cz}\mu(dz) \leq 2$. Then by the result of Willmot and Lin, for any $r > 0$, and n large enough $\mathrm{P}\left(Z_{t_{k+1}} - Z_{t_k} \geq r\right) \leq 2e^{-cr}$.

Theorem

For all $(\theta, \beta) \in \Theta \times \Sigma$ *and* $w = (u, v) \in \mathbb{R}^2$, *as* $n \to \infty$,

$$\log \frac{p(X^n; (\theta_n, \beta_n))}{p(X^n; (\theta, \beta))} \xrightarrow{\mathcal{L}(\mathrm{P}_x^{\theta,\beta})} w^{\mathsf{T}} \mathcal{N}(0, \Gamma(\theta, \beta)) - \frac{1}{2} w^{\mathsf{T}} \Gamma(\theta, \beta) w,$$

where $\theta_n = \theta + \frac{u}{\sqrt{n\Delta_n}}$, $\beta_n = \beta + \frac{v}{\sqrt{n}}$, *and* $\mathcal{N}(0, \Gamma(\theta, \beta))$ *is a centered* $\mathbb{R}^2$*-valued Gaussian random variable with covariance matrix*

$$\Gamma(\theta, \beta) = \begin{pmatrix} \int_{\mathbb{R}} \left(\frac{\partial_\theta b(\theta, x)}{\sigma(\beta, x)} \right)^2 \pi_{\theta,\beta}(dx) & 0 \\ 0 & 2 \int_{\mathbb{R}} \left(\frac{\partial_\beta \sigma(\beta, x)}{\sigma(\beta, x)} \right)^2 \pi_{\theta,\beta}(dx) \end{pmatrix}.$$

Work in progress

1. The case of infinite jumps requires a convergence argument.

2. The case where the drift is unbounded and we have a general Poisson random measure :

$$dX_t = b(\theta, X_t)dt + \sigma(\beta, X_t)dB_t + \int_{\mathbb{R}_0} c(X_{t-}, z)\,(N(dt, dz) - \nu(dz)dt)\,.$$

References

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