

Laws Relating Runs and Steps in Gambler's Ruin

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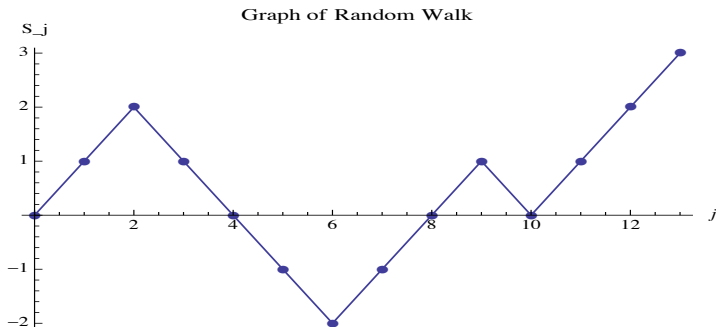
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Simple Random Walk.

Simple Random Walk on the Integers: $\mathbf{S}_j, j = 0, 1, 2, \dots$

- Independent fair coin tosses determine the steps.
- Example:

$\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2, \dots = 0, 1, 2, 1, 0, -1, -2, -1, 0, 1, 0, 1, 2, 3 \dots$



Excursion: Length, Runs, Height

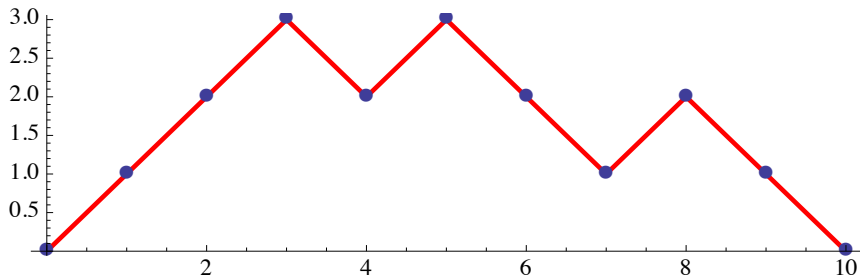
- Excursion:

A Path of First Return to Zero by Random Walk

Example Shows :

Number of Runs: $R = 6$

Length: $L = 10$; Height: $H = 3$



Narayana Polynomials

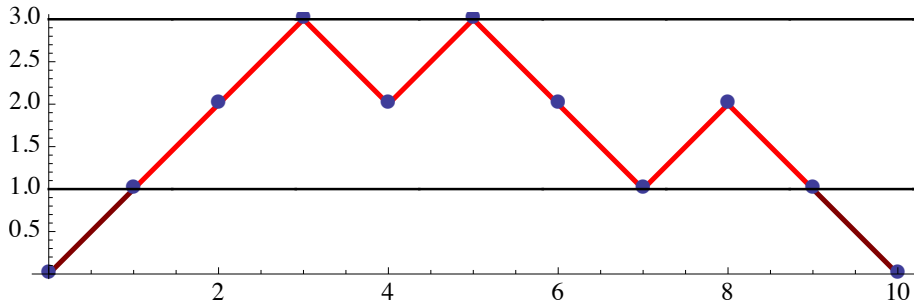
- **n -Dyck Path** with k **Peaks**: Non-negative, Length $2n$, Zero at Ends

Counted by:
$$\mathcal{N}_{n,k} := \binom{n}{k-1} \binom{n-1}{k-1} / k$$

- **Narayana Polynomial:**

$$\mathcal{N}_n(s) := \sum_{k=1}^n \mathcal{N}_{n,k} s^{k-1}; \quad \mathcal{N}_1(s) = 1.$$

$$\mathcal{N}_2(s) = 1 + s, \quad \mathcal{N}_3(s) = 1 + 3s + s^2, \quad \mathcal{N}_4(s) = 1 + 6s + 6s^2 + s^3.$$



Joint G. F. of Runs and Length: $\rho = \frac{1}{2}$

By raising a $(n-1)$ -**Dyck Path** to Level 1,

$$P_{n,k} := P(\mathbf{R} = 2k, \mathbf{L} = 2n) = 2 \times 4^{-n} \mathcal{N}_{n-1,k}, \quad P_{1,1} = \frac{1}{2}.$$

Therefore

$$\text{(G.F.) } E\{y^{\mathbf{R}} z^{\mathbf{L}}\} = \sum_{n=1}^{\infty} \sum_{k=1}^n P_{n,k} y^{2k} z^{2n} = \frac{y^2 z^2}{2} \left(1 + \sum_{n=1}^{\infty} \mathcal{N}_n(y^2) (z^2/4)^n \right).$$

Break up

$$\mathcal{N} := 1 + \sum_{n=1}^{\infty} \mathcal{N}_n(s) t^n$$

Into Pure Excursions Plus Dyck Paths with At Least One "Middle" Zero.

$$\implies \mathcal{N} = 1 + t\mathcal{N} + st(\mathcal{N}^2 - \mathcal{N})$$

\implies **Fact (Joint G.F. of Runs and Steps):**

- $E\{y^{\mathbf{R}} z^{\mathbf{L}}\} = 1 + xu - \sqrt{(xu - 1)^2 - 4x}, \quad u := y^2 - 1, \quad x := z^2/4.$

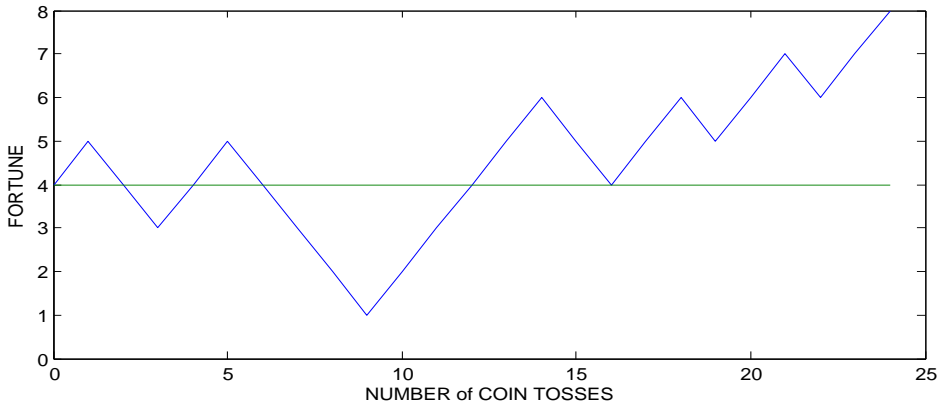
Example for Gambler's Ruin

Initial Fortune N . Play Independent Fair Games: Win 1 or Lose 1.

Terminate Game when **Fortune** First Equals $2N$ or 0.

Equivalently, Terminate at Epoch: $\inf\{j : |\mathbf{S}_j| = N\}$.

In Example, the "Last Visit" is: $\mathcal{L} = 16$, with $N = 4$.



Goal: Explicitly Find $K_N(y, z) := E\{y^{\mathbf{R}} z^{\mathbf{L}} | \mathbf{H} < N\}$.

- Let $\mathbf{L}^{(1)}, \mathbf{L}^{(2)}, \dots$ be Independent Copies of \mathbf{L} .
- Similarly $\mathbf{R}^{(1)}, \mathbf{R}^{(2)}, \dots$ are Independent Copies of \mathbf{R}
- Let $\mathcal{M} = \mathcal{M}_N := \#$ Excursions Until the Last Visit \mathcal{L} .
- \mathcal{M} is a Geometric random variable:

$$P(\mathcal{M} = m) = P(\mathbf{H} < N)^m P(\mathbf{H} \geq N), \quad m = 0, 1, 2, \dots$$

- Write: $K_N(y, z) := E\{y^{\mathbf{R}} z^{\mathbf{L}} | \mathbf{H} < N\}$,

$$\mathcal{L} := \sum_{m=0}^{\mathcal{M}} \mathbf{L}^{(m)}, \quad \mathcal{R} := \sum_{m=0}^{\mathcal{M}} \mathbf{R}^{(m)}.$$

- Hence, **Application:**

$$E\{y^{\mathcal{R}} z^{\mathcal{L}}\} = \sum_{m=0}^{\infty} P(\mathcal{M} = m) (K_N(y, z))^m = \frac{P(\mathbf{H} \geq N)}{1 - P(\mathbf{H} < N) K_N(y, z)}.$$

Non-increasing Future Maxima in an Excursion

Future Maxima for Excursion Shown:

$M_1 :=$ Max. Height Over Excursion. Here, $M_1 = 4$.

$M_2 :=$ Max. Height **After** Return from M_1 to Level 1.

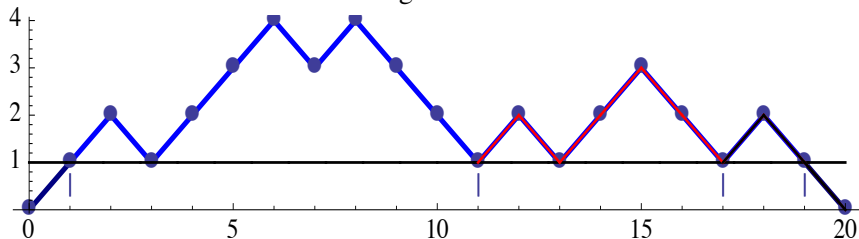
Here, $M_2 = 3$, $M_3 = 2$.

Within Segment i , the Path Forms:

(a) A Non-negative First Passage from Level 1 to M_i ,

(b) An Upside-Down Non-negative First Passage.

Excursion with 3 Segments for Future Maxima



Conditional G. F. of \mathbf{R} and \mathbf{L} Given $\mathbf{H} = N$

- $\mathbf{L}_N := \#$ Steps Until a **First Passage** to $\mathbf{H} = N$.
- $\mathbf{R}_N := \#$ Runs Until a **First Passage** to $\mathbf{H} = N$.
- Conditional Joint G. F. of \mathbf{R}_N and \mathbf{L}_N ,
Given **Non-negative Passage** to $\mathbf{H} = N$:

$$g_N(y, z) := E\{y^{\mathbf{R}_N} z^{\mathbf{L}_N} \mid \mathbf{S}_j \geq 0, j = 1, \dots, \mathbf{L}_N\}.$$

- $zg_N(y, z)$ Is the Joint Distribution of Runs and Steps over **Gambler's Ruin Meander**.
- By **Non-negative Passage** to level N from Level 1, Followed by an **Upside Down Non-negative Passage** to Level $N+1$, the Conditional G. F. of \mathbf{R} and \mathbf{L} Given $\mathbf{H} = N + 1$ is:

$$(1) G_{N+1}(y, z) := E\{y^{\mathbf{R}} z^{\mathbf{L}} \mid \mathbf{H} = N + 1\} = zg_N g_{N+1}.$$

Recurrence Formula for $G_N(y, z)$.

- Define

$$\rho_N := P(\mathbf{S}_j \geq 0, j = 1, \dots, \mathbf{L}_N).$$

- By Classical Gambler's Ruin Solution, if $\rho = \frac{1}{2}$, $\rho_N = \frac{1}{N+1}$.
- **Future Maxima Decomposition** \implies

$$G_{N+1} = Cz^2 \sum_{\ell=1}^{\infty} (\rho_N g_N)^{2\ell} \prod_{a=1}^{N-1} \sum_{\ell_a=0}^{\infty} (\rho_a g_a)^{2\ell_a},$$

$$[\ell := \max\{i : M_1 = \dots = M_i = N + 1\}.]$$

$$\implies (2) \quad G_{N+1} = Cz^2 g_N^2 \prod_{a=1}^N \frac{1}{1 - [\rho_a g_a]^2}.$$

- (1) and (2) \implies

$$(3) \quad g_{N+1} = z^{-1} g_N^{-1} G_{N+1} = Cz g_N \prod_{a=1}^N \frac{1}{1 - [\rho_a g_a]^2}.$$

Recursion for g_N .

- Equation (3) may be Rewritten:

$$(3)' \quad g_{N+1} = C \frac{g_N}{g_{N-1}} \frac{1}{1 - [\rho_N g_N]^2} \left(z g_{N-1} \prod_{a=1}^{N-1} \frac{1}{1 - [\rho_a g_a]^2} \right).$$

Lemma

$$(*) : \quad g_{N+1} = C_N \frac{g_N^2}{g_{N-1}} * \frac{1}{1 - [\rho_N g_N]^2}, \quad N \geq 2,$$

where, if $p = \frac{1}{2}$, $C_N = 1 - \rho_N^2 = \frac{(N+2)N}{(N+1)^2}$, and $g_1 = yz$, $g_2 = \frac{3}{4} \frac{yz^2}{1 - y^2 z^2 / 4}$.

- For $p = \frac{1}{2}$, Write **Denominator** w_2 of g_2 by

$$w_2 = 1 - x(u + 1), \text{ for } x := \frac{1}{4} z^2, \quad u := y^2 - 1.$$

Denominators w_N of g_N

- **Introduce** $u := y^2 - 1$ and $x := \xi z^2$, for $\xi := p(1 - p)$.

- For $p = \frac{1}{2}$, **Lemma** $(*) \implies w_2 = (1 - x) - ux$,

$$w_3 = (1 - 2x) - (2x - x^2)u + x^2u^2,$$

$$w_4 = (1 - 3x + x^2) - (3x - 4x^2)u + (3x^2 - x^3)u^2 - x^3u^3.$$

- In General, **Denominators** w_N of g_N Satisfy the **Identity**:

$$(**) \quad w_N^2 - x^N(u + 1) = w_{N-1}w_{N+1}; \quad w_0 = w_1 = 1.$$

- For $p = \frac{1}{2}$, and $u = 0$: $w_N = N^{\text{th}}$ Fibonacci Polynomial, F_N .

$$F_{N+1} = F_N - xF_{N-1}.$$

- Via Mathematica and OEIS, **Guess General Formula** for w_N .

Denominators w_N

Define

$$w_N(x, u) := \sum_{j \geq 0} \sum_{k \geq 0} \binom{N-k-j}{k} \binom{N-k-1}{j} (-x)^k (-ux)^j, \quad N \geq 1.$$

Lemma

$$\sum_{N=1}^{\infty} w_N(x, u) t^N = \frac{t(1-xt)}{1+(xu-1)t+xt^2}.$$

- **Lemma** \implies Closed Form for w_N :

$$w_N = \frac{2^{-N}}{\alpha} \{(\beta + \alpha)^N - (\beta - \alpha)^N - 2x[(\beta + \alpha)^{N-1} - (\beta - \alpha)^{N-1}]\},$$

$$\beta := 1 - xu, \quad \alpha := \sqrt{(xu - 1)^2 - 4x}.$$

- **Identity** (**) for Denominators Established Algebraically By Applying **Closed Form**.

Numerators $q_N(u)$

- Define **Numerators**:

$$q_N := \frac{2^{-N}}{\alpha} \{(\beta + \alpha)^N - (\beta - \alpha)^N\}$$

- Commutation Identity** Follows Algebraically from Closed Forms:

$$(***) \quad q_N w_{N+1} - w_N q_{N+1} = -x^N.$$

Theorem

$$\frac{C}{y^2 z^2} E\{y^R z^L \mid \mathbf{H} \leq N\} = \sum_{a=1}^N G_a P(\mathbf{H} = a) = \sum_{a=1}^N \frac{x^{a-1}}{w_{a-1} w_a} = \frac{q_N}{w_N}.$$

- Proof of Theorem: $(***) \implies \frac{q_N}{w_N} + \frac{x^N}{w_N w_{N+1}} = \frac{q_{N+1}}{w_{N+1}}$.

Corollary

$$E\{y^R z^L \mid \mathbf{H} < \infty\} = \frac{1 + \sqrt{1 - 4\xi}}{4\xi} (1 + xu - \sqrt{(xu - 1)^2 - 4x}).$$

Distribution of Runs.

Corollary

If $p = \frac{1}{2}$, then

$$P(\mathbf{R} = 2n) = \frac{1}{\pi} \int_0^\pi \frac{\sin^2(\theta) d\theta}{(5 + 4 \cos \theta)^{n+1}}, \quad n \geq 2; \quad P(\mathbf{R} = 2) = 2/3.$$

- Proof: Expand $q_N(\frac{1}{4}, u)/w_N(\frac{1}{4}, u)$ by Partial Fractions.

Application to Gambler's Ruin: $p = \frac{1}{2}$

- $\mathcal{R} = \mathcal{R}_N :=$ Sum of Runs over Excursions in Gambler's Ruin.

- Formula $E\{y^{\mathcal{R}} z^{\mathcal{L}}\} = \frac{P(\mathbf{H} \geq N)}{1 - P(\mathbf{H} < N)K_N(y, z)}$, $z = 1$, and $y = e^{-\lambda/N^2} \implies$

Corollary

$$\lim_{N \rightarrow \infty} E\{e^{-\lambda \mathcal{R}_N / N^2}\} = \frac{\tanh \sqrt{\lambda}}{\sqrt{\lambda}} = \int_0^\infty f(x) e^{-\lambda x} dx,$$

$$f(x) = (\pi x)^{-1/2} \sum_{\nu=-\infty}^{\infty} (-1)^\nu e^{-\nu^2/x}, \quad x > 0.$$

Scaling Limit for $2\mathcal{R} - \mathcal{L}$ in Gambler's Ruin: $p = \frac{1}{2}$.

- $\mathcal{L}_N :=$ # Steps over Excursions in Gambler's Ruin = "Last Visit".

$$\lim_{N \rightarrow \infty} E\{e^{-\lambda \mathcal{L}_N / N^2}\} = \frac{\tanh \sqrt{2\lambda}}{\sqrt{2\lambda}}$$

- Expect $2\mathcal{R}_N - \mathcal{L}_N$ to have order N .
- Denote

$\mathcal{R}'_N :=$ #Runs over Meander, $\mathcal{L}'_N :=$ #Steps over Meander,

$$X_N := 2\mathcal{R}_N - \mathcal{L}_N - \mathcal{M}_N, \quad X'_N := 2\mathcal{R}'_N - \mathcal{L}'_N.$$

Corollary

$$\lim_{N \rightarrow \infty} E\{e^{it(X_N + X'_N)/N}\} = \frac{1}{\cosh(t)} = \int_{-\infty}^{\infty} e^{itx} s_1(x) dx,$$

$$\lim_{N \rightarrow \infty} E\{e^{itX'_N/N}\} = \frac{\sinh(t)}{t} = \int_{-\infty}^{\infty} e^{itx} s_2(x) dx,$$

$$s_1(x) = \frac{1}{2} \operatorname{sech}(\pi x / 2), \quad s_2(x) = \frac{\pi}{4} \operatorname{sech}^2(\pi x / 2), \quad -\infty < x < \infty.$$

Integer Sequences

From OEIS A059231,

- $a(n) := \#$ Lattice Paths $(0, 0)$ to $(2n, 0)$ using steps from

$$S = \{(k, k) \text{ or } (k, -k) : k \geq 1\}, \text{ and never crossing the } x\text{-axis.}$$

- Paths Counted by $a(n)$ are so-called "Catalan Rook Paths".
- $a(n) = 1, 5, 29, 185, 1257, \dots$
- "Twin" Generating Function to $K(y) := E\{y^{\mathbf{R}}\}$:

$$A(t) := \sum_{n=0}^{\infty} a(n)t^n = \frac{1}{8t} \left(1 + 3t - \sqrt{1 - 10t + 9t^2} \right) \quad [\text{Kung + de Mier}]$$

$$\text{Relationship: } K(y) = 2A(1/y^2).$$

Corollary

$$a(n) = \frac{1}{2} 3^{2n+1} P(\mathbf{R} = 2(n+1)), \quad n \geq 0.$$

- The **Future Maxima Decomposition** is amenable to the study of the conditional Joint Generating Function of Runs and Steps given the Height of a Random Walk Excursion.
- **Applications:**
 - A $(\text{Sech})^2$ -type Scaling Distribution Holds for Twice the Number of Runs Minus the Number of Steps in the Gambler's Ruin Meander.
 - A Binomial Identity Arises from the Denominator Polynomials $w_N(x, u)$ in the Special Case $x = \frac{1}{4}$.
- **Outlook**
 - Understand the Relationship Between "Catalan Rook Paths" Enumerated by $a(n) = 1, 5, 29, \dots$, and the Distribution of Runs.

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Alternative Formulae for w_N and Q_N with $x = \frac{1}{4}$.

- Special Case: $x = \frac{1}{4}$.

$$w_N\left(\frac{1}{4}, u\right) = 2^{-N} \sum_{j=0}^{N-1} \left[\binom{N+j}{2j+1} + \binom{N+j-1}{2j} \right] \left(\frac{-u}{2}\right)^j.$$

- Note Form of $q_N\left(\frac{1}{4}, u\right)$:

$$q_N\left(\frac{1}{4}, u\right) := 2^{-N+1} \sum_{j=0}^{N-1} \binom{N+j}{2j+1} \left(\frac{-u}{2}\right)^j.$$

- Binomial Identity Arising from Two Representations of $w_N\left(\frac{1}{4}, u\right)$:

$$\begin{aligned} & \sum_{k \geq 0} \binom{N-k-j}{k} \binom{N-k-1}{j} \left(-\frac{1}{4}\right)^k \\ &= 2^{j-N} \left(\binom{N+j}{2j+1} + \binom{N+j-1}{2j} \right), \quad \text{all } j \geq 0, N \geq 1. \end{aligned}$$