Variational formula for the time-constant of first passage percolation

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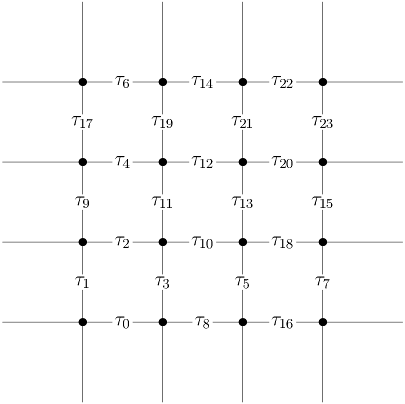
First Passage Percolation on the Lattice



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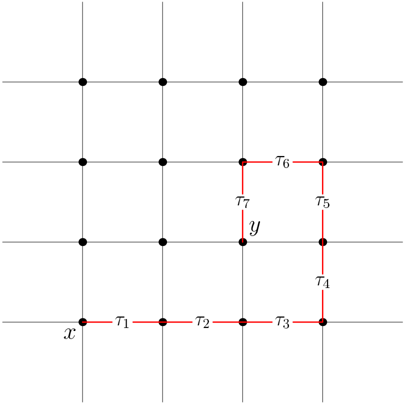

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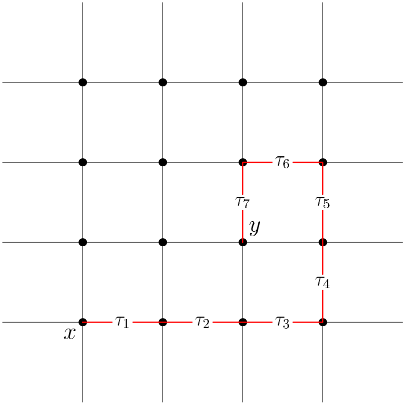

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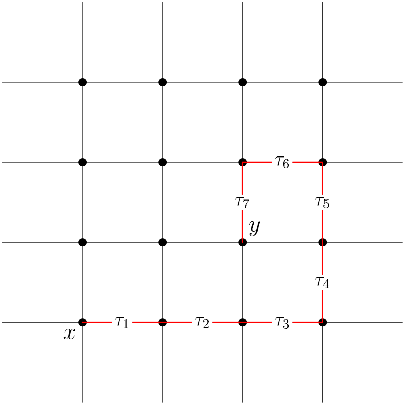

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- Will write $T(x)$ for $T(x, 0)$ in general

What do we want to compute?

Time-constant $g(x)$

- Fix $x \in \mathbb{R}^d$, consider an "average" time to travel in direction x.

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- $g(x)$ is called time-constant.

Motivation: the limit-shape

(or invasion cluster)

Consider sites occupied by time t:

$$R_t := \{x \in \mathbb{R}^d \mid T([x]) \leq t\},$$

We're interested in the limiting behavior of this set (shape theorem of Cox and Durrett (1981)).

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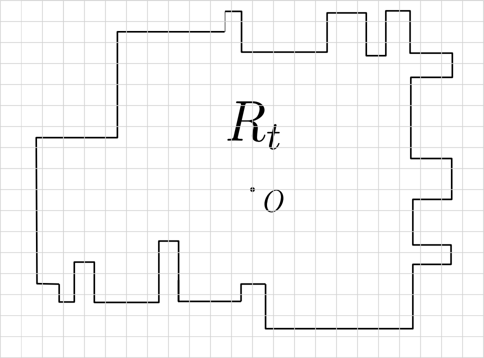

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- Movement of light in a medium: Eikonal equation.

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- $g(x)$ is a norm on $\mathbb{R}^d$
- Solve PDE: $H(p)$ is the dual norm

  $$H(p) = \sup_{g(x)=1} x \cdot p$$

Outline

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- Future work/other applications

Notation for main theorem

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- $\tau(z, \cdot, \omega)$ is stationary-ergodic (e.g. i.i.d.), and edge-weights are uniformly bounded: $0 < a \leq \tau \leq b$.
- For $f : \mathbb{Z}^d \to \mathbb{R}$, discrete derivative is $Df(x, \alpha) = f(x + \alpha) - f(x)$.

Main Theorem

Variational Formula

Theorem

For $p \in \mathbb{R}^d$, the dual norm of $g(x)$ is given by

$$H(p) = \inf_{f \in S} \operatorname*{ess\,sup}_{\omega \in \Omega} \mathcal{H}(Df + p, x, \omega),$$

where

$\mathcal{H}$ is the discrete Hamiltonian

S is a set of functions.

Link:algorithm

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where

$$\mathcal{H}(Df + p, x, \omega) = \sup_{\alpha \in A} \left\{ -\frac{Df(x,\alpha) + p \cdot \alpha}{\tau(x,\alpha,\omega)} \right\},$$

$$S = \left\{ f : \mathbb{Z}^d \to \mathbb{R} \mid E[Df] = 0, Df \text{ stationary} \right\}.$$

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- Replace this with a single variational problem for $H(p)$; minimization over functions
- Think of this is a nonlinear duality principle

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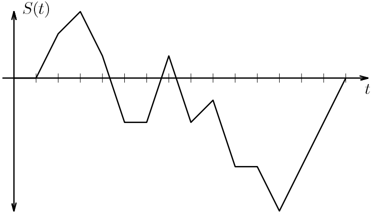
- Polymers and last-passage percolation models: concurrent results of Georgiou, Rassoul-Agha, and Seppäläinen (2013). Same formula, different approach.

Why should $g(x)$ satisfy a PDE?

Analogy with Brownian motion

$S(t)$ linearly interpolated version of simple random walk with ±1 increments.



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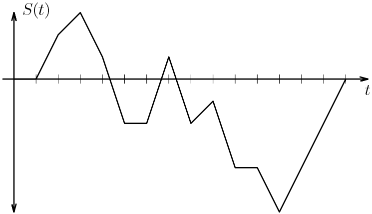

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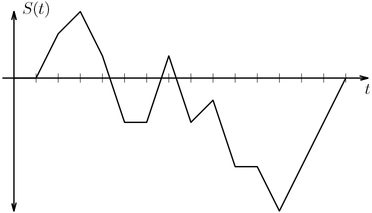

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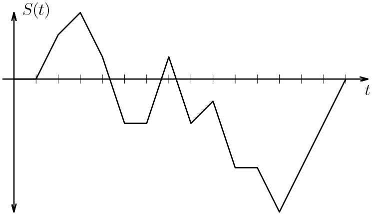

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Imagine we don't know what $B(t)$ is.

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- $T_n(x) = T([nx])/n$ is scaled first-passage time, $g(x)$ time-constant
- By subadditive ergodic theorem $T_n(x) \to g(x)$
- Dynamic programming principle
$$T(x) = \inf_{\pm\alpha\in\mathcal{A}}\{T(x+\alpha)+\tau(x,\alpha)\}.$$
- Want to show
$$H(Dg(x)) = 1$$

Application

Exact limit-shape by iteration

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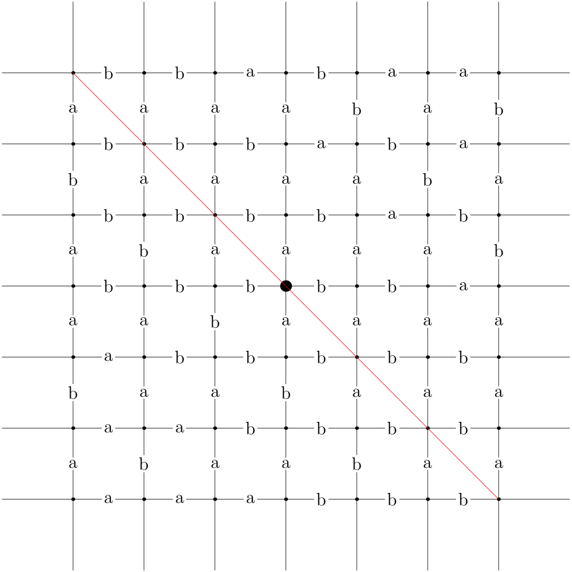
Symmetry Assumption

For each $z \in \mathbb{Z}$, assume

$$\tau(x, \cdot, \omega) = \tau(y, \cdot, \omega) \quad \forall\, x + y = z.$$

Line of symmetry

$\tau(\cdot,\cdot,\omega) \in \{a, b\}, a < b$



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Link:variational formula

Theorem: constructing the minimizer

For any $f_0 \in S$, we give an explicit $I : S \to S$ such that the sequence defined by

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Proof implies

One of the following happens:

- Algorithm terminates in finite time at a corrector
- Algorithm terminates in finite-time at a generic minimizer
- Algorithm continues to infinity, produces corrector in limit

Examples of limit shapes

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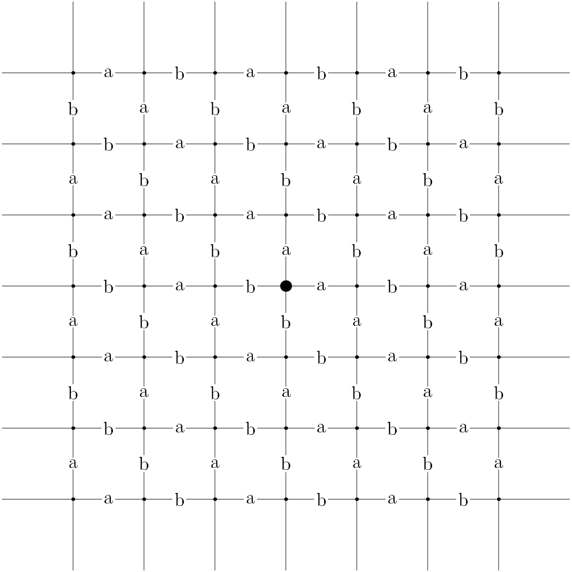
- Will play around with edge-weight marginals; all supported on $[1, 2]$.
- Will see the level sets $\{p \in \mathbb{R}^2 : H(p) = 1\}$ in dimension $d = 2$.
- The "bigger" the Hamiltonian level-set, the slower the percolation. It's a speed-time duality.

Example: Periodic Medium

$\tau(\cdot,\cdot,\omega) \in \{a, b\}, a < b$



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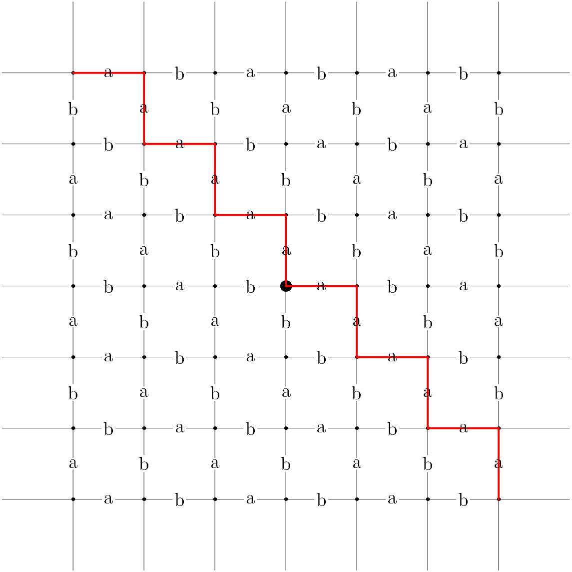
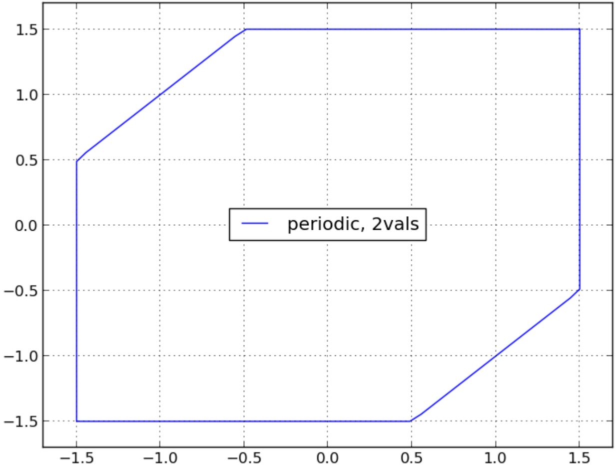


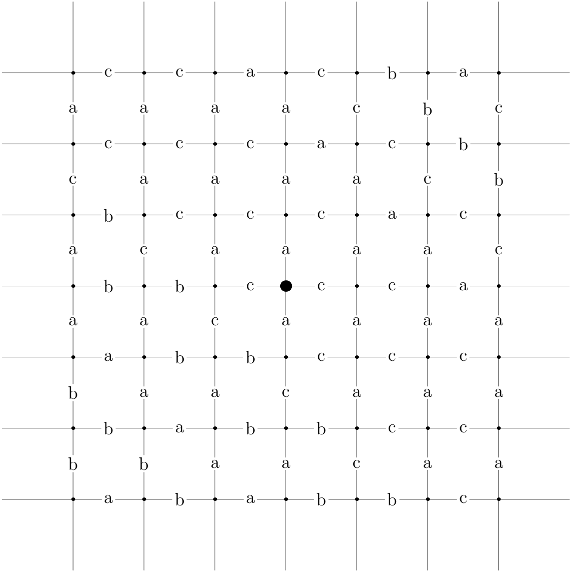
Limit Shape: Periodic Medium

$\tau \in \{1, 2\}$, Plot of $H(p) = 1$



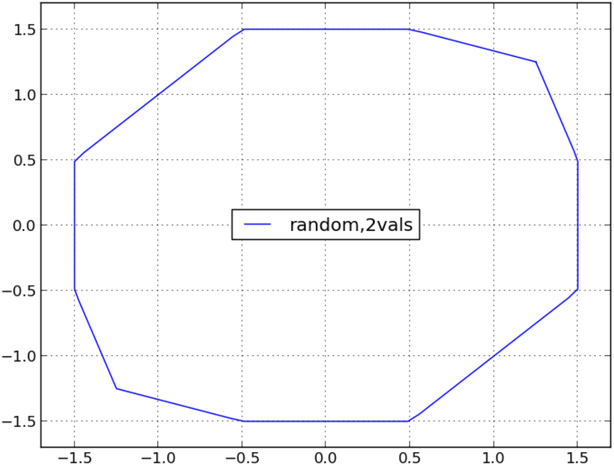
Example: Random Medium

$\tau(\cdot,\cdot,\omega) \in \{a,b,c\}$ $\tau(\cdot,\cdot,\omega) \in \{a,b,c\}$, $a < c < b$



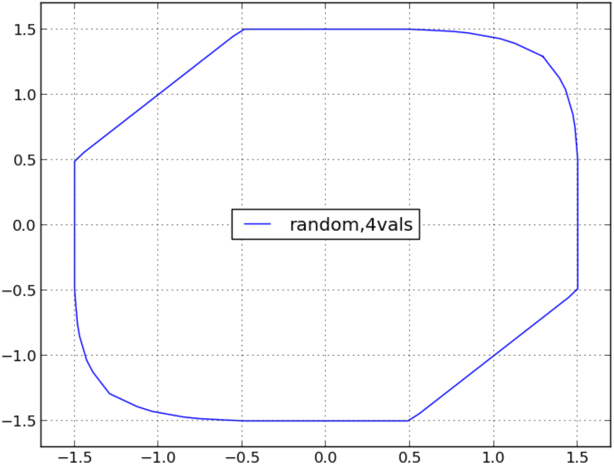
Limit Shape: Random Medium

$\tau \in \{1, 2\}$, uniform measure, plot of $H(p) = 1$



Limit Shape: Random Medium

$\tau \in \{1, 4/3, 5/3, 2\}$, uniform measure, plot of $H(p) = 1$



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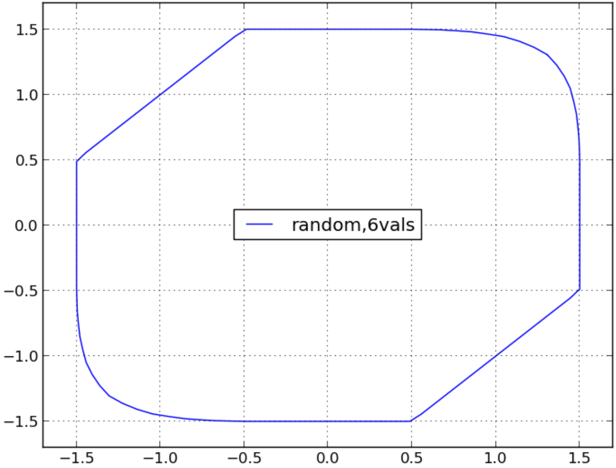
$\tau \in \{1, 6/5, 7/5/, 8/5, 9/5, 2\}$, uniform measure, plot of $H(p) = 1$



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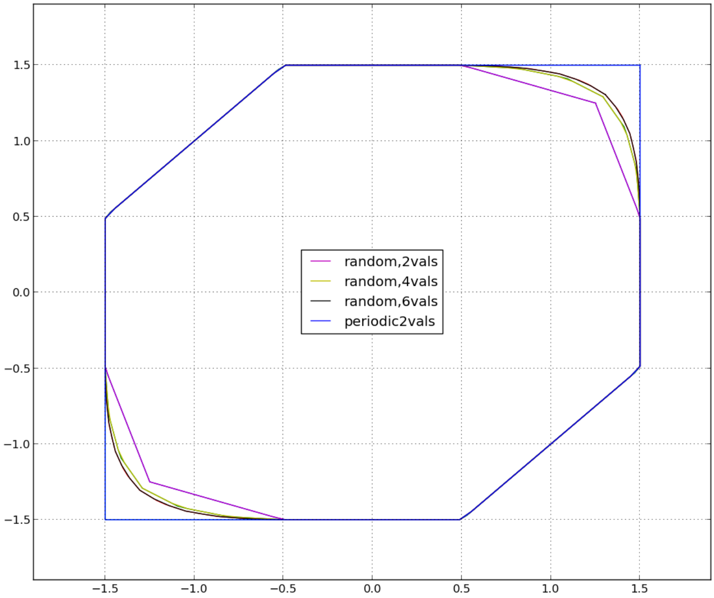

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- Strict convexity of $H(p) \Leftrightarrow$ regularity of $g(x)$.
  Can the iteration be directly used to prove existence of correctors, uniqueness of minimizer and hence strict convexity of $H(p)$? (cf. Auffinger and Damron (2013))

Future Work/Open Questions

Iteration and Regularity

- Upgraded full iteration without symmetry assumption: believe it's possible if you can handle "percolating maxima".
- Strict convexity of $H(p) \Leftrightarrow$ regularity of $g(x)$.
  Can the iteration be directly used to prove existence of correctors, uniqueness of minimizer and hence strict convexity of $H(p)$? (cf. Auffinger and Damron (2013))
- I believe this is possible for monotone Hamiltonians (directed first-passage percolation, polymer models).