# Fractals at infinity and SPDEs
# (Large scale random fractals)

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# (local) Hausdorff dimension

Let $E \subset \mathbf{R}^d$. For every $\alpha \geqslant 0$ and $\delta > 0$, define

$$\mathcal{H}^\alpha_\delta(E) := \inf \left\{ \sum |E_i|^\alpha ; E \subset \bigcup_{i \geqslant 1} E_i, |E_i| < \delta \right\}$$

and then

$$\mathcal{H}^\alpha(E) := \lim_{\delta \searrow 0} \mathcal{H}^\alpha_\delta.$$

Then the (local) Hausdorff dimension of the set $E$ is defined as

$$\dim_H(E) := \sup\{\alpha; \mathcal{H}^\alpha(E) = \infty\} = \inf\{\alpha; \mathcal{H}^\alpha(E) < \infty\}.$$

- the Cantor set, the range of Brownian motion, the zero set of the Brownian motion, the set of fast times ($\{t \in [0,1]; \limsup_{h \searrow 0} |B(t+h) - B(t)|/\sqrt{2h \log(1/h)} \geqslant \gamma\}$, and so on.
- Q) sets on $\mathbf{Z}^d$??
- Q) sets at infinity, e.g. $\{t \in \mathbf{R}^+; B(t) \geqslant \sqrt{2\gamma t \log\log t}\}$ (the set of points of peaks)??

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## Large scale Hausdorff dimension by Barlow and Taylor ('88 and '91)

Let $\mathcal{V}_n := [-e^{n-1}, e^{n-1})^d$, $\mathcal{S}_1 := \mathcal{V}_1$, $\mathcal{S}_{n+1} := \mathcal{V}_{n+1} \setminus \mathcal{V}_n$ for all $n \geqslant 1$.
For every $\alpha > 0$, define $\nu_\alpha^n$ on each $\mathcal{S}_n$ as

$$\nu_\alpha^n(E, \mathcal{S}_n) := \inf \left\{ \sum_{i=1}^{m} \left( \frac{|Q_i|}{e^n} \right)^\alpha ; E \cap \mathcal{S}_n \subset \bigcup_{i=1}^{m} Q_i, 1 \leqslant |Q_i| \leqslant e^n \right\}.$$

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- If $A \subset B$, then $\mathrm{Dim}_H(A) \leqslant \mathrm{Dim}_H(B)$.
- $\mathrm{Dim}_H(\mathbf{Z}^k) = k$.
- $\mathrm{Dim}_H$(the range of the simple random walk on $\mathbf{Z}^d$) $= d \wedge 2$.
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# Peaks of Brownian motion

Let $B(t)$ be the standard Brownian motion on **R**. Consider the following set of exceedance times (peaks): for $\gamma > 0$

$$\mathcal{L}_B(\gamma) := \left\{ t \geqslant e^e : \; B(t) \geqslant \sqrt{2\gamma t \log\log t} \right\}.$$

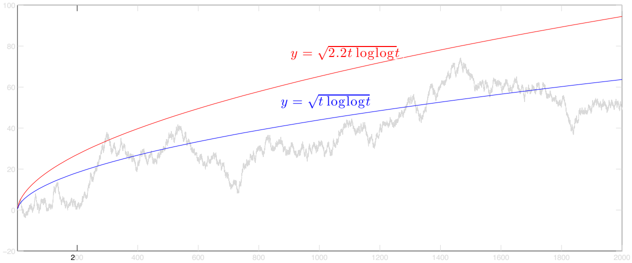


Figure : the red line shows $\mathcal{L}_B(1.1)$ and the blue line shows $\mathcal{L}_B(0.5)$.

- [Law of the Iterated Logarithm]
  - $\mathcal{L}_B(\gamma)$ is unbounded a.s. when $\gamma \leqslant 1$;
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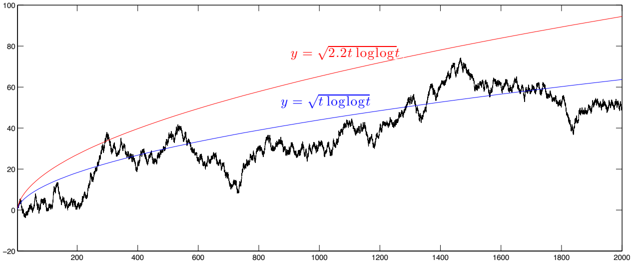


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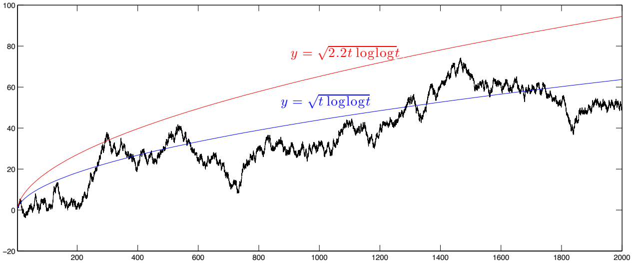


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# Peaks of Brownian motion

**Definition 1 (Asymptotic Density)**

$$\mathrm{Den}(E) := \limsup_{t \to \infty} \frac{|E \cap [0, t]|}{t}.$$

**Theorem 2 (Strassen '64)**

*For $\gamma \in (0, 1)$,*

$$\mathrm{Den}\left(\mathcal{L}_B(\gamma)\right) = 1 - \exp\left(-4(\gamma^{-1} - 1)\right).$$

**Theorem 3 (Khoshnevisan–K–Xiao)**

*Almost surely,*

$$\mathrm{Dim}_H(\mathcal{L}_B(\gamma)) = \begin{cases} 1, & \text{if } 0 < \gamma \leqslant 1 \\ 0, & \text{if } \gamma > 1. \end{cases}$$

- $\mathrm{Dim}_H(\mathcal{L}_B(\gamma))$ is always 0 or 1.
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# Peaks of the Ornstein-Uhlenbeck process

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$$U(t) := \frac{B(e^t)}{e^{t/2}},$$
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- $U(t)$ is centered Gaussian with $\mathrm{E}[U(t)U(s)] = \exp(-|t-s|/2)$ (O-U process).
- (LIL) $\limsup_{t\to\infty} \frac{U(t)}{\sqrt{2\log t}} = 1, \quad a.s.$
- $\mathcal{L}_U(\gamma) = \log \mathcal{L}_B(\gamma)$, thus $\mathcal{L}_U(\gamma)$ is unbounded a.s. iff $\gamma \leqslant 1$.

**Theorem 4 (Khoshnevisan–K–Xiao)**

*If* $\gamma \in (0, 1]$, *then* $\mathrm{Dim}_H(\mathcal{L}_U(\gamma)) = 1 - \gamma$ *a.s.*

- Recall, for a Brownian motion, $\mathrm{Dim}_H(\mathcal{L}_B(\gamma)) = 1$ for $0 < \gamma \leqslant 1$.
- Brownian motion is *mono-fractal* whereas the Ornstein-Uhlenbeck process is *multi-fractal*.
- $\mathrm{Dim}_H\, \mathcal{L}_U(1) = 0$ but $\mathrm{Dim}_H \exp(\mathcal{L}_U(1)) (= \mathcal{L}_B(1)) = 1$. (cf. $\mathrm{Dim}_H(\mathbf{N}) = 1$ but $\mathrm{Dim}_H(\exp(\mathbf{N})) = 0$.)

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# High peaks in SPDEs (space-time white noise)

- Linear stochastic heat equation

$$\dot{z}_t(x) = \frac{1}{2} z_t''(x) + \xi_t(x) \quad x \in \mathbf{R}, t > 0, \quad \text{and} \quad z_0(x) = 0.$$

- $z_t(x)$ is centered Gaussian.
- $\limsup_{x\to\infty} z_t(x)/\sqrt{2\log x} = (t/\pi)^{1/4}$ a.s.
- Let $\mathcal{L}_{z_t}(\gamma) := \left\{ x \geqslant \mathrm{e} : \; z_t(x) \geqslant \left(\frac{t}{\pi}\right)^{1/4} \sqrt{2\gamma \log x} \right\}$ $\quad (t, \gamma > 0)$.
- No intermittency ($\mathrm{E} z_t^2(x) = \sqrt{t/\pi}$, i.e. the moment Lyapunov exponent is 0)

- Stochastic parabolic anderson model

$$\dot{u}_t(x) = \frac{1}{2} u_t''(x) + u_t(x)\xi_t(x) \quad x \in \mathbf{R}, t > 0, \quad \text{and} \quad u_0(x) = 1.$$

- Let $h_t(x) := \log u_t(x)$. This $h_t(x)$ is the Hopf-Cole solution of the KPZ equation.
- $0 < \limsup_{x\to\infty} h_t(x)/(\log x)^{2/3} < \infty$ a.s. (Conus-Joseph-Khoshnevisan '13)
- Let $\mathcal{L}_{h_t}(\gamma) := \left\{ x \geqslant \mathrm{e} : h_t(x) \geqslant \gamma\, t^{1/3} (\log x)^{2/3} \right\}$.
- Intermittency ($\mathrm{E}[u_t(x)]^k \approx e^{ctk^3}$, i.e. the moment Lyapunov exponent is like $k^3$).

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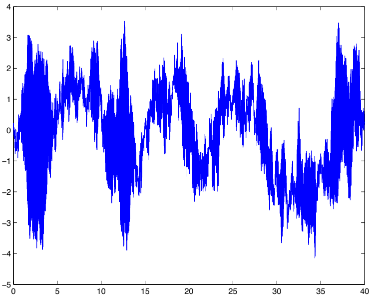
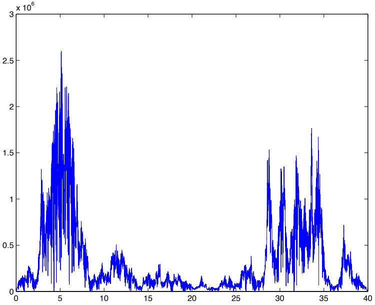
# Simulations




(a) Linear stochastic heat equaiton




(b) Parabolic Anderson model

Q) How about the large scale dimensions of the exceedance sets $\mathcal{L}_{z_t}(\gamma)$ and $\mathcal{L}_{h_t}(\gamma)$?

Recall $\mathcal{L}_{z_t}(\gamma) := \left\{ x \geqslant \mathrm{e} : \; z_t(x) \geqslant \left(\frac{t}{\pi}\right)^{1/4} \sqrt{2\gamma \log x} \right\} \qquad (t, \gamma > 0).$

**Theorem 5 (Khoshnevisan–K.–Xiao)**

*Choose and fix $t > 0$. The set $\mathcal{L}_{z_t}(\gamma)$ is almost surely unbounded if $\gamma \leqslant 1$; else, if $\gamma > 1$ then $\mathcal{L}_{z_t}(\gamma)$ is almost surely bounded. Furthermore,*

$$\mathrm{Dim}\, \mathcal{L}_{z_t}(\gamma) = 1 - \gamma,$$

*a.s. for all $t > 0$ and $\gamma \in (0\,, 1]$.*

Recall $\mathcal{L}_{h_t}(\gamma) := \left\{ x \geqslant \mathrm{e} : h_t(x) \geqslant \gamma\, t^{1/3} (\log x)^{2/3} \right\}$ and $h_t(x) = \log u_t(x)$.

**Theorem 6 (Khoshnevisan–K.–Xiao)**

*For every $t$ and $\gamma > 0$, there exist two constants $0 < \alpha \leqslant \beta < \infty$ such that*

$$1 - \beta\gamma^{3/2} \leqslant \mathrm{Dim}\, \mathcal{L}_{h_t}(\gamma) \leqslant 1 - \alpha\gamma^{3/2} \qquad \textit{a.s.}$$

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**Theorem 6 (Khoshnevisan–K.–Xiao)**

*For every $t$ and $\gamma > 0$, there exist two constants $0 < \alpha \leqslant \beta < \infty$ such that*

$$1 - \beta\gamma^{3/2} \leqslant \mathrm{Dim}\, \mathcal{L}_{h_t}(\gamma) \leqslant 1 - \alpha\gamma^{3/2} \qquad \textit{a.s.}$$

# General Bound (Upper bound)

Let $X := \{X_t\}_{t \geqslant 0}$ be a stochastic process with values in **R**. Define

$$\boldsymbol{c}(b) := -\limsup_{z\to\infty} z^{-b} \sup_{t\geqslant 0} \log \mathrm{P}\{X_t > z\},$$
$$\boldsymbol{C}(b) := -\liminf_{z\to\infty} z^{-b} \inf_{t\geqslant 0} \log \mathrm{P}\{X_t > z\}.$$

**Theorem 7 (Upper bound)**

*Suppose that there exists $b \in (0\,,\infty)$ such that $\boldsymbol{c}(b) > 0$ and for all $\gamma \in (0\,,1)$,*

$$\sup_{w\geqslant 0} \mathrm{P}\left\{ \sup_{t\in[w,w+1]} X_t > \left( \frac{\gamma}{\boldsymbol{c}(b)} \log s \right)^{1/b} \right\} \leqslant s^{-\gamma+o(1)} \text{ as } s \to \infty.$$

*Then,* $\limsup_{t\to\infty} (\log t)^{-1/b} X_t \leqslant [\boldsymbol{c}(b)]^{-1/b}$ *a.s. Furthermore,*

$$\mathrm{Dim}_H \left\{ t > \mathrm{e} : \; X_t \geqslant \left( \frac{\gamma}{\boldsymbol{c}(b)} \log t \right)^{1/b} \right\} \leqslant 1 - \gamma,$$

*for all $\gamma \in (0\,,1)$.*

# General Bound (Lower bound)

### Definition 8

For every $n \geqslant 1$ and $\delta \in (0\,,1)$, let $\mathcal{G}_n(\delta)$ denote the collection of all finite sequences $\mathrm{e}^n \leqslant t_1 < t_2 < \ldots < t_m < \mathrm{e}^{n+1}$—in $\mathcal{S}_n$—such that $t_{i+1} - t_i \geqslant \exp(\delta n)$ for all $1 \leqslant i \leqslant m$.

### Definition 9

Let $\mathcal{I}$ denote the collection of all independent finite sequences of independent random variables.

### Theorem 10 (Lower bound)

*Suppose there exists $b \in (0\,,\infty)$ such that $\boldsymbol{C}(b) < \infty$.*

$$n^{-1} \max_{\{t_i\}_{i=1}^m \in \mathcal{G}_n(\delta)} \max_{1 \leqslant j \leqslant m} \inf_{\{Y_i\}_{i=1}^m \in \mathcal{I}} \log \mathrm{P}\{|X_{t_j} - Y_j| > 1\} \to -\infty,$$

*as $n \to \infty$. Then,* $\limsup_{t\to\infty} (\log t)^{-1/b} X_t \geqslant [\boldsymbol{C}(b)]^{-1/b}$ *a.s. Moreover, if $\gamma \in (0\,,1)$ then*

$$\mathrm{Dim}\left\{ t > \mathrm{e} :\ X_t \geqslant \left( \frac{\gamma}{\boldsymbol{C}(b)} \log t \right)^{1/b} \right\} \geqslant 1 - \gamma \text{ a.s.}$$

Thank You!