Brownian Motion on Lie Groups: Limits and Fluctuations in Large Dimensions

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Giving Credit where Credit is Due

Based partly on joint work with Bruce Driver (UC San Diego), Brian Hall (Notre Dame), and Guillaume Cébron (Université Paris 6).

- Driver; Hall; K: *The large-N limit of the Segal–Bargmann transform on $\mathbb{U}_N$*. J. Funct. Anal. 265, 2585–2644 (2013)
- K: *Heat kernel empirical laws on $\mathbb{U}_N$ and $\mathbb{GL}_N$*. To appear in J. Theor. Probab. arXiv:1306.2140.
- K: *The Large-N Limits of Brownian Motions on $\mathbb{GL}_N$*. arXiv:1306.6033.
- Cébron; K: *Fluctuations of Brownian Motions on $\mathbb{GL}_N$*. Preprint in preparation.

Brownian Motion on Hermitian Matrices: the Gaussian Unitary Ensemble

The Gaussian Unitary Ensemble (GUE) and its Empirical Spectral Distribution (ESD)

Fix an infinite double-array $\{B_{jk}(t)\colon\, j,k \geq 1\}$ of i.i.d. $\mathbb{C}$-valued Brownian motions. The GUE_N-valued Brownian motion $X^N(t)$ is the matrix process

$$[X^N(t)]_{jk} = \frac{1}{\sqrt{N}}\left(B_{jk}(t) + \overline{B_{kj}(t)}\right).$$

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Denote $\lambda_1^N(t), \lambda_2^N(t), \ldots, \lambda_N^N(t)$ the (random) eigenvalues of $X^N(t)$. The **empirical spectral distribution** ESD μ_t^N is the random measure

$$\mu_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{\lambda_j^N(t)}.$$

Wigner's Law (for fixed $t > 0$)

The first theorem of Random Matrix Theory was proved by Wigner; the following is a stronger version.

Theorem. (Wigner, 1955) For fixed $t > 0$, the ESD μ_t^N converges weakly almost surely to the **semicircle law**

$$\sigma_t(dx) = \frac{1}{2\pi t}\sqrt{(4t - x^2)_+}\, dx.$$

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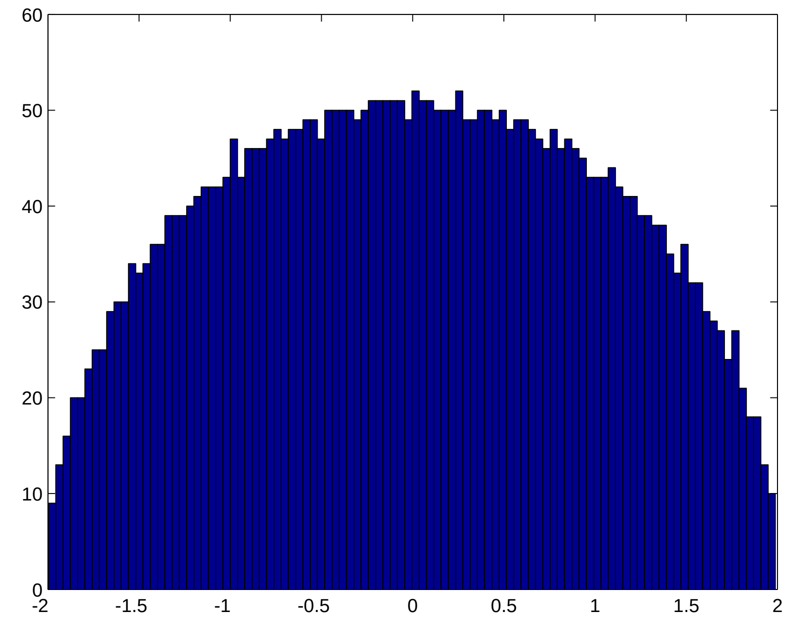

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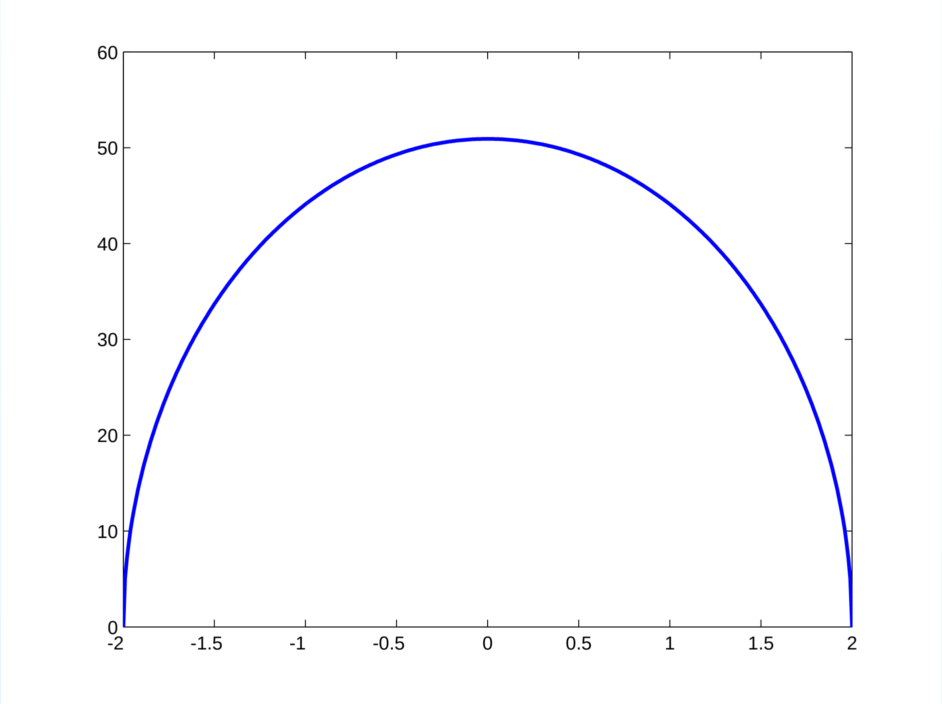

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Notice that

$$\int f\, d\mu_t^N = \frac{1}{N}\sum_{j=1}^{N} f(\lambda_j^N(t)) = \mathrm{tr}[f(X^N(t))].\ \ (\mathrm{tr} = \frac{1}{N}\mathrm{Tr})$$

Taking $f =$ polynomials allows combinatorial techniques (the method of moments), which is how Wigner proved the theorem (which holds essentially regardless of the distribution of entries).

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One can gain a lot of mileage from this representation in deciphering, e.g., the asymptotic distribution of the spacing between the eigenvalues. But it turns out to be less useful for understanding the limit and fluctuation of the measure-valued ESD process μ_t^N.

Smooth Drift and Propagation of Chaos

Suppose $\alpha\colon \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz. Let $B_1(t), \ldots, B_N(t)$ be i.i.d. standard Brownian motions. Define the processes $\eta_j^N(t)$ by the SDE

$$d\eta_j^N = \frac{1}{\sqrt{N}} dB_j(t) + \frac{1}{N} \sum_{k \neq j} \alpha(\eta_j^N(t), \eta_k^N(t))dt, \qquad 1 \le j \le N.$$

Let $\nu_t^N = \frac{1}{N} \sum_{j=1}^N \delta_{\eta_j^N(t)}$.

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Theorem. (McKean, 1967) There is a measure ν_t so that $\nu_t^N \to \nu$ weakly in probability, and ν_t is the law of an Itô process.

Theorem. (Hitsuda & Mitoma, 1986) The fluctuations of ν_t^N are $O(\frac{1}{\sqrt{N}})$, and $\phi_t^N \equiv \sqrt{N}(\nu_t^N - \nu_t)$ converges weakly to a Gaussian process.

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Since the drift α in Dyson's Brownian motion is not Lipschitz, the theorems do not apply. In fact, they fail spectacularly!

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Free probability is a field at the intersection of probability theory, complex analysis, and operator algebras. The basic idea is to represent limits of random matrix ensembles like $X^N(t)$ as $N \to \infty$ as concrete objects. The *free* comes from the fact that the limits often live in an operator algebra modeled on a free group.

Convergence in (Noncommutative) Probability

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Given a vector $\mathbf{A}^N = (A_1^N, \ldots, A_m^N)$ of $N \times N$ random matrices, and a vector $\mathbf{a} = (a_1, \ldots, a_m) \in \mathscr{A}^m$, we say $\mathbf{A}^N \xrightarrow{\mathscr{D}} \mathbf{a}$ if

$$\mathrm{tr}(P(\mathbf{A})) \to \tau(P(\mathbf{a})) \text{ for any } \textit{noncommutative} \text{ polynomial } P.$$

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Theorem. (Voiculescu, 1991) In the free group factor $(\Lambda(\mathbb{F}_\infty), \tau)$, there is a "process" $(x_t)_{t\ge 0}$ such that, for any $t_1, \ldots, t_m \ge 0$,

$$(X^N(t_1), \ldots, X^N(t_m)) \xrightarrow{\mathscr{D}} (x_{t_1}, \ldots, x_{t_m}).$$

The process $(x_t)_{t\ge 0}$ is known as the **free (additive) Brownian motion**. Its increments are **freely** independent, and the spectral measure of $x_t - x_s$ is σ_{t-s} for $s \le t$.

Free Stochastic Differential Equations

Let $\mathscr{A}_t \subset \Lambda(\mathbb{F}_\infty)$ denote the (weakly closed) subalgebra generated by $(x_s)_{s\le t}$. Call a process $(\theta_t)_{t\ge 0}$ **adapted** if $\theta_t \in \mathscr{A}_t$ for each $t \ge 0$.

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In the 1990s, Biane and Speicher showed that SDEs like

$$d\theta_t = \alpha(t, \theta_t)\, dx_t + \beta(t, \theta_t)\, dt$$

have unique adapted solutions with given initial data in $\mathscr{A}_0$, provided the diffusion and drift coefficients α, β are sufficiently smooth and slow-growing (the proof uses the same Picard iteration method that works classically).

Heat Kernels and Brownian Motion on Lie Groups

The Laplacian Δ on a Riemannian Manifold

Let M be a Riemannian manifold, with metric $\langle\cdot,\cdot\rangle$ and resulting volume form dV. If $f \in C^\infty(M)$, the **gradient** $\nabla f = \nabla_M f$ is the vector field defined by

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If U is an isometry of M, then $(\Delta f)\circ U = \Delta(f\circ U)$. This means Δ can be computed by the same expression in any orthonormal basis. If $M = \mathbb{R}^n$ with its usual Euclidean metric, $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$.

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$$\rho(t,x,y) = (2\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{2t}}.$$

This Gaussian tail behavior is universal; but in general there is no formula for the heat kernel on any non-Euclidean manifold.

Left-Invariant Laplacian on a Lie Group

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- Since $e^{\frac{t}{2}\Delta_G}(1) = 1$, ρ_t is a probability density.
- Since $e^{\frac{s+t}{2}\Delta_G} = e^{\frac{s}{2}\Delta_G} e^{\frac{t}{2}\Delta_G}$, $\rho_{s+t} = \rho_s * \rho_t$.

We will also denote by $d\rho_t$ the **heat kernel measure** (with density ρ_t). This measure is determined (by definition) by

$$\int_G f\,d\rho_t = \left(e^{\frac{t}{2}\Delta_G} f\right)(1_G), \qquad f \in C_c(G).$$

Brownian Motion on Riemannian Manifolds, and Lie Groups

The Brownian motion $B_t^{x_0}$ on a Riemannian manifold M is the Markov process with generator $\frac{1}{2}\Delta_M$, started at $x_0 \in M$.

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- $t \mapsto B_t$ is a continuous map from $\mathbb{R}_+$ into G a.s.
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There is an even more explicit representation, as a kind of projection of the Brownian motion on the Lie algebra. Let β be an o.n. basis of $\mathfrak{g}$, and

$$W_t = \sum_{\xi\in\beta} W_t^{(\xi)}\xi, \quad \{W_t^{(\xi)}\}_{\xi\in\beta} \text{ i.i.d. Brownian motions on } \mathbb{R}.$$

Then, in Stratonovich form, $\quad dB_t = B_t \circ dW_t$.

Brownian Motions on $\mathfrak{u}_N$ and $\mathfrak{gl}_N$

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As a (real) orthonormal basis of $\mathfrak{gl}_N$, we can take the matrix units $\{\frac{1}{\sqrt{N}} E_{jk}\}_{1 \le j,k \le n} \cup \{\frac{i}{\sqrt{N}} E_{jk}\}_{1 \le j,k \le n}$, and so the Brownian motion $Z^N(t)$ can be written as

$$[Z^N(t)]_{jk} = \frac{1}{\sqrt{N}} [W_{jk}(t) + iW'_{jk}(t)]$$

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It is a routine exercise to find an o.n. basis for $\mathfrak{u}_N$, and find that the Brownian motion there has the form $-iX^N(t)$, where

$$X^N(t) = \frac{1}{2}[Z^N(t) + Z^N(t)^*].$$

Brownian Motions on $\mathbb{U}_N$ and $\mathbb{GL}_N$

There is a general procedure for converting Stratonovich integrals to Itô integrals. In the case of $\mathfrak{g}$-valued Brownian motion W_t, this gives the Itô SDE

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In particular, $\sum_{\xi\in\beta} \xi^2 = -I$. Similarly, $\beta' = \beta \cup i\beta$ is an o.n. basis for $\mathfrak{gl}_N$, and so it follows that $\sum_{\xi\in\beta'} \xi^2 = -I - (i^2)I = 0$.

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$$dU_t = iU_t\, dX_t - \frac{1}{2} U_t\, dt, \qquad dB_t = B_t\, dZ_t.$$

Large-N Limits of Brownian Motions on $\mathfrak{u}_N$, $\mathfrak{gl}_N$, $\mathbb{U}_N$, and $\mathbb{GL}_N$

Large-N Limits of Free Additive Brownian Motion

Theorem. [Voiculescu, 1991] Let $X^N(t)$ and $Z^N(t)$ be the Brownian motions on $\mathfrak{u}_N$ and $\mathfrak{gl}_N$. Then, for any times $t_1, \ldots, t_m \geq 0$,

$$(X^N(t_1), \ldots, X^N(t_m)) \xrightarrow{\mathscr{D}} (x_{t_1}, \ldots, x_{t_m}) \; a.s., \quad \text{and}$$

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where $z_t = \frac{1}{\sqrt{2}}(x_t + iy_t)$, and x_t, y_t are freely independent free additive Brownian motions. Note: in the complex case, $\mathscr{D}*$ convergence means that for any noncommutative polynomial P in in $2m$ variables,

$$\lim_{N\to\infty} \operatorname{tr}[P(Z^N(t_1), Z^N(t_1)^*, \ldots, Z^N(t_n), Z^N(t_m)^*)]$$
$$= \tau[P(z_{t_1}, z_{t_1}^*, \ldots, z_{t_m}, z_{t_m}^*)].$$

Free Stochastic Differential Equations

Using the free stochastic calculus developed by Biane and Speicher, one can solve the free versions of the matrix SDEs defining the Brownian motions on $\mathbb{U}_N$ and $\mathbb{GL}_N$.

$$du_t = iu_t\,dx_t - \frac{1}{2}u_t\,dt$$
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It is now natural to ask whether the same kind of convergence of processes $U_t^N \to u_t$ and $B_t^N \to b_t$ holds true. The unitary case was solved by Biane in 1997; the general linear case took 16 years longer.

Free Unitary Brownian Motion

Theorem. [Biane, 1997] Let U_t^N be the Brownian motion on $\mathbb{U}_N$, and let u_t be a free unitary Brownian motion, defined by $du_t = iu_t\,dx_t - \frac{1}{2}u_t\,dt$. Then for any times $t_1, \ldots, t_m \geq 0$,

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As a corollary, this shows that the ESD of U_t^N converges a.s. to a deterministic measure ν_t on the circle $\mathbb{U}_1$, whose moments can be computed explicitly:

$$\int_{\mathbb{U}_1} \theta^n \nu_t(d\theta) = \tau(u_t^n) = e^{-\frac{n}{2}t} \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} n^{k-1} \binom{n}{k+1}$$

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constants that had come up before in Singer's 1991 paper "The Master Field on the Plane". The measure ν_t is symmetric about 1, has a smooth density in its support. It is compactly supported strictly in $\mathbb{U}_1$ for $t < 2$, and fully supported for $t \geq 2$.

Free Multiplicative Brownian Motion

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Theorem. [K, 2013] Let B_t^N be Brownian motion on $\mathbb{GL}_N$, and let b_t be a free multiplicative Brownian motion, defined by $db_t = b_t\, dz_t$. Then for any times $t_1, \ldots, t_m \geq 0$,

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My approach is more geometric, involving a more direct calculation of the involved moments, using the description of the Laplacian above. The key idea is to find the right space of functions (that include moment functions) invariant under $\Delta_{\mathbb{GL}_N}$.

Trace Polynomials and their Intertwining Space

An Example Calculation

Example. Consider the function $f(A) = \mathrm{tr}(A^2A^*)$ on $\mathbb{GL}_N$. We use the "magic formulas"

$$\sum_{\xi\in\beta_N} \xi A\xi = -\mathrm{tr}(A)I_N, \quad \sum_{\xi\in\beta_N} \mathrm{tr}(A\xi)\xi = -\frac{1}{N^2}A.$$

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This 2×2 system can be exponentiated by a (good) freshman, and we see that

$$e^{\frac{t}{2}\Delta_{\mathbb{GL}_N}} f = e^{2t}\cosh(2t/N)f + e^{2t}N\sinh(2t/N)g$$

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$$\begin{aligned}e^{\frac{t}{2}\Delta_{\mathbb{GL}_N}} f &= e^{2t}\cosh(2t/N)f + e^{2t}N\sinh(2t/N)g \\ &= e^{2t}f + 2te^{2t}g + O(1/N^2).\end{aligned}$$

(Abstract) Trace Polynomial Space

Let $\mathscr{P}$ denote the commutative $\mathbb{C}$-algebra generated by the set of finite words $\varepsilon \in \bigcup_{n=0}^{\infty}\{1,*\}^n$. For convenience, label the basis elements v_ε. For example

$$P = v_1 - 2v_{1*1} + v_{*1}v_1.$$

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Call such elements **abstract trace polynomials**. The reason is the following. For any $N \in \mathbb{N}$ and any $P \in \mathscr{P}$, define a function $P_N\colon \mathbb{M}_N \to \mathbb{C}$ as follows: for any word $\varepsilon = \varepsilon_1 \cdots \varepsilon_n$, let

$$[v_\varepsilon]_N(A) = \mathrm{tr}(A^{\varepsilon_1} \cdots A^{\varepsilon_n})$$

then extend the map $P \mapsto P_N$ to be an algebra homomorphism. For example, with the above P,

$$P_N(A) = \mathrm{tr}(A) - 2\mathrm{tr}(AA^*A) + \mathrm{tr}(A^*A)\mathrm{tr}(A).$$

Such functions are called **trace polynomials**.

Intertwining Formula for the Laplacian

Theorem. [Driver, Hall, K.] The space $[\mathscr{P}]_N$ of trace polynomials is a reducing subspace for $\Delta_{\mathbb{GL}_N}$. There exist first- and second-order differential operators $\mathcal{D}$ and $\mathcal{L}$ on $\mathscr{P}$ so that

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The point is that $e^{\frac{t}{2}\left(\mathcal{D}+\frac{1}{N^2}\mathcal{L}\right)} = e^{\frac{t}{2}\mathcal{D}} + O\left(\frac{1}{N^2}\right)$. Since $e^{\frac{t}{2}\mathcal{D}}$ is an algebra homomorphism, this leads to the following core estimate.

Corollary. For any trace polynomials P, Q,

$$\mathrm{Cov}(P_N(B_t^N), Q_N(B_t^N)) = O\left(\frac{1}{N^2}\right).$$

Idea of the Proof of Convergence $(B_t^N)_{t\geq 0} \to (b_t)_{t\geq 0}$

Fix $t > 0$. We have $dB_t = B_t\, dZ_t$ and $db_t = b_t\, dz_t$. Because the diffusion terms are linear, we can proceed by induction on the degree of the moment. Using stochastic calculus, the difference can be expressed as an integral of terms consisting of the difference between lower-order moments (which $\to 0$ by inductive hypothesis), plus the covariance of the involved terms (which $\to 0$ as above).

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Fluctuations of Matrix Brownian Motions

Fluctuations of Flat Brownian Motions

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For the "flat" Brownian motions $X^N(t)$ and $Z^N(t)$, this was answered by Speicher and Mingo in 2006, and later in 2009 in slightly greater generality by Chatterji in 2009, for the case of "linear statistics".

Theorem. Let $p_1, \ldots, p_m$ be polynomials in one variable. Let $t_1, \ldots, t_m \geq 0$. Then the random variables

$$N[\mathrm{tr}(p_j(X^N(t_j))) - \mathbb{E}\mathrm{tr}(p_j(X^N(t_j)))], \qquad j = 1 \ldots m$$

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A similar result (involving polynomials in the variables and their adjoints) holds for $Z^N(t)$.

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Note, since $U^N(t)$ is a normal matrix, $f(U^N(t))$ can be made sense of for any measurable function f on the unit circle, via functional calculus. But this doesn't allow for multiple times, since that introduces real noncommutativity.

Fluctuations of Brownian Motion on $\mathbb{U}_N$

A partial answer to the fluctuations question for unitary Brownian motion was given by Lévy and Maïda in 2010.

Theorem. [Lévy, Maïda, 2010] Fix a time $t > 0$. Let $f_1, \ldots, f_m$ be Lipschitz functions. Then the random variables

$$N[\mathrm{tr}(f_j(U^N(t))) - \mathbb{E}\mathrm{tr}(f_j(U^N(t)))], \qquad j = 1 \ldots m$$

are, in the limit as $N \to \infty$, jointly Gaussian, with a covariance determined by $f_1, \ldots, f_m$.

Note, since $U^N(t)$ is a normal matrix, $f(U^N(t))$ can be made sense of for any measurable function f on the unit circle, via functional calculus. But this doesn't allow for multiple times, since that introduces real noncommutativity. Using a different approach, we can handle the more general $\mathbb{U}_N$ case, and the $\mathbb{GL}_N$ case, simultaneously.

Fluctuations of Brownian Motion on $\mathbb{GL}_N$

Theorem. [Cébron, K, 2014] Let $P_1, \ldots, P_n$ be trace polynomials. Let $t_1, \ldots, t_m \geq 0$. Let $\Xi^N(t)$ denote either $U^N(t)$ or $B^N(t)$. Then the random variables

$$X_j = N[P_j(\Xi^N(t_1), \ldots, \Xi^N(t_m)) - \mathbb{E}P_j(\Xi^N(t_1), \ldots, \Xi^N(t_m))]$$

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Recall that $\Delta_{\mathbb{GL}_N} \sim \mathcal{D} + \frac{1}{N^2}\mathcal{L}$. The fluctuations are therefore controlled by the second-order operator $\mathcal{L}$; in fact, by its *carré du champ* operator

$$\Gamma(P, Q) = \mathcal{L}(PQ) - \mathcal{L}(P)Q - P\mathcal{L}(Q).$$

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Indeed, we can express the covariance of the asymptotically Gaussian random vector $(X_1, \ldots, X_n)$ as follows:

The Covariance of the Fluctuations of B_t^N (for a fixed t)

Theorem. [Cébron, K, 2014] The asymptotic covariance matrix of $(X_1, \ldots, X_n)$ has (j,k)-entry $\sigma(P_j, P_k)$, where the function σ is determined as follows: given $P, Q \in \mathscr{P}$, there is a trace polynomial $\tilde{\Gamma}(P,Q)$ in three variables such that, if a_t, b_t, c_t are three freely independent multiplicative Brownian motions,

$$\sigma(P,Q) = \int_0^t \left[\tilde{\Gamma}(P,Q)\right](a_s, b_{t-s}, c_{t-s})\, ds.$$

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E.g. Suppose p, q are single-variable polynomials. Then

$$\sigma(\mathrm{tr}(p), \mathrm{tr}(q^*)) = \int_0^t \tau\left[p'(b_{t-s}a_s)q'(c_{t-s}a_s)^*\right]\, ds.$$

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In the unitary case, we can compute that this converges (as $t \to \infty$) to $\langle p, q\rangle_{H_{1/2}}$, agreeing with [Diaconis, Evans, 2001] in the Haar unitary case.

A More Complete Story

As a final remark: *much* of the technology here applies more generally than the Brownian motion. In the papers cited, everything was done for a two-parameter family of Brownian motions (corresponding to all unitarily-invariant inner products on $\mathbb{GL}_N$); but, in fact, similar techniques yield similar results for more general Lévy processes and diffusion processes on $\mathbb{GL}_N$ (and subgroups). ○