Threshold models for rainfall and convection: deterministic and stochastic triggers

Scott Hottovy
shottovy@math.wisc.edu
www.math.wisc.edu/∼shottovy

Work with Sam Stechmann (UW Math)

University of Wisconsin

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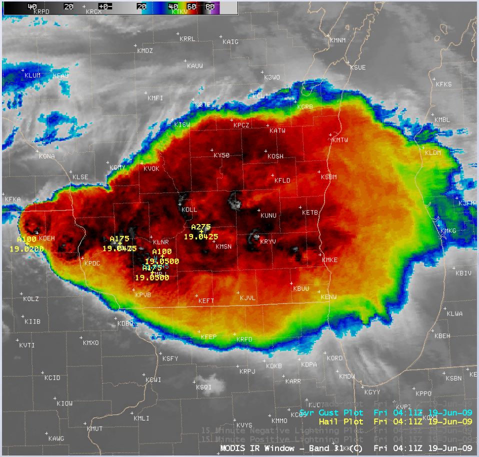
Mesoscale Convective Systems

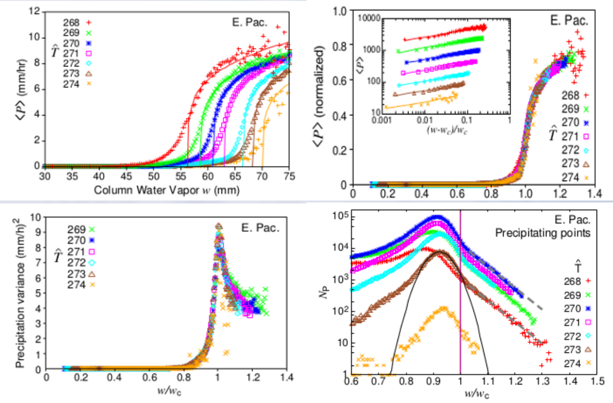


Example of a MCS over Wisconsin, 5.6" rain in Dodgeville (19/06/2009). (cims.ssec.wisc.edu)

Transition to Strong Convection

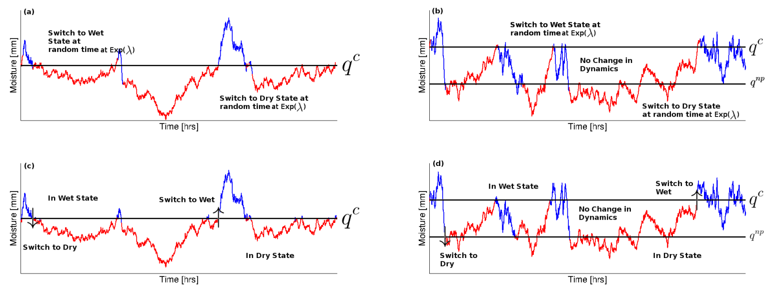
- Water vapor accumulates in the atmosphere, then transition to strong convection and precipitation. Water vapor then decreases until a level where convection then is “turned off.”
- Goals:
  1. Model moisture, in a single cloud, with a trigger(s) that indicate precipitation (strong convection).
  2. Introduce 4 simple models of column water vapor (moisture) with different triggers.
  3. Compare and contrast the models with aim of producing statistics that mirror observational data.
  4. Show a simplified model is “close” to a more complicated model that mimics the obsv stats.
  5. (Future:) Use simple model to make a complicated multi-cloud model.
  6. (Future:) Study possible phase transitions.

Observational Data



From: J.D. Neelin, O. Peters, K. Hales: The transition to strong convection. J. Atmos. Sci., 66, 2367-2384, doi:10.1175/2009JAS2962.1 (2009).

Four models



Clockwise, (S1) (S2), (D2), (D1).

$$dq_t = \begin{cases} mdt + D_0 dW_t, & \sigma_t = 0 \\ -rdt + D_1 dW_t, & \sigma_t = 1 \end{cases} \qquad D_1 > D_0 > 0$$

A Note on the (D1) Model

- For $q^{np} = q^c$, single threshold cases:
- (S1) is similar to (S2).
- (D1) is much more different than (D2),

$$dq_t = \begin{cases} -r dt + D_1\, dW_t & q \geq q^c \\ m dt + D_0\, dW_t & q < q^c \end{cases}$$

- Referred to as Filippov or Hysterisis in Dynamical Systems.
- Recently called sliding dynamics (Kuske-Simpson 2012, 2013)
- For $D_0 = D_1 = 1$, studied by Karatzas and Shreve (1984).

Stationary Densities

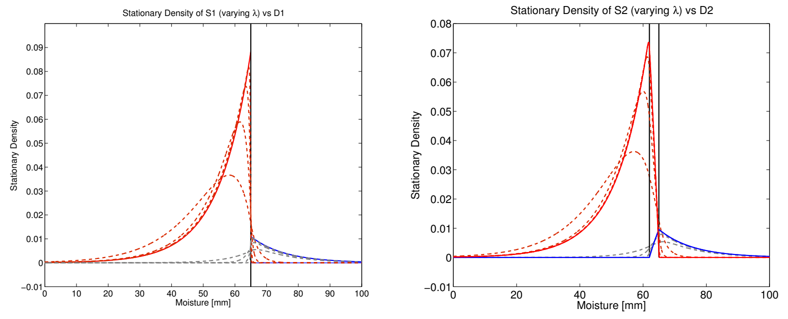
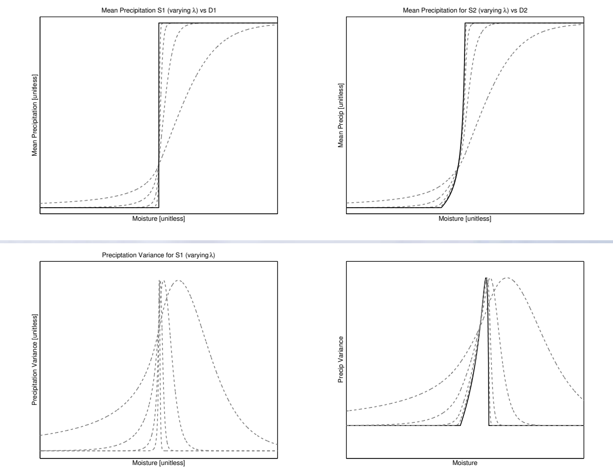


Figure : Plot of the stationary densities $\rho_0(q)$ (red) and $\rho_1(q)$ (blue) for deterministic (solid) and stochastic (dashed) triggers for various values of $\lambda^{-1} = 4, 0.4, .04, .004$ hours. The vertical dashed lines are the cut offs q^{np} (black) and q_c (grey).

Model Statistics



Mean precip. $E[\sigma|q]$. Deterministic (Black) and Stochastic (Grey) triggers, with 1 (left) and 2 (right) thresholds.

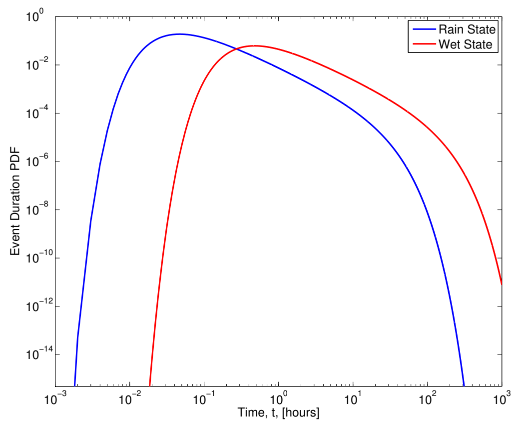
Other Interesting Statistics

- Cloud Fraction.
  - In steady state,

$$E[\sigma] = \frac{r}{m+r}. \tag{1}$$

    Independent of # of thresholds and stochastic vs deterministic trigger.
- Event Size.
  - Transient: What is the probability an individual rain event drops R mm of rain?
  - Related to probability a rain event lasts t hours.
  - For (D2), just the first passage time for reaching level $q^c - q^{np} = q^\epsilon$.

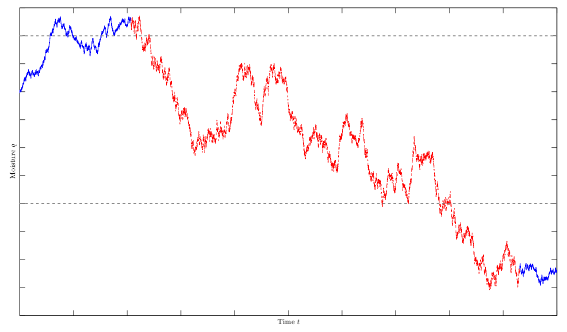
Event Duration for (D2)



Question: How does event duration scale in λ with (S2)?

Event Duration

For stochastic trigger, not so simple.



Event duration τ_{ET} = First passage time from for Brownian motion with drift $x + q^{\epsilon}$ units, then random jumping time τ^{j}.

$$\tau_{ET} = \tau_{FP} + \tau^{J}$$

Using Laplace Transform

Consider the time-dependent Fokker-Planck equation,

$$\begin{cases} \partial_t \rho_0^J = -m\partial_q \rho_0^J + \frac{D_0^2}{2}\partial_q^2 \rho_0^J - \lambda H(q-q^c)\rho_0^J \\ \partial_t \rho_1^J = \lambda H(q-q^c)\rho_0^J, \\ \rho_0^J(q,0) = \delta(q-q^c) \quad \rho_1^J(q,0) = 0 \end{cases}$$

Taking the Laplace transform,

$$s\tilde{\rho}_0^J - \delta(q-q^c) = -m\partial_q \tilde{\rho}_0^J + \frac{D_0^2}{2}\partial_q^2 \tilde{\rho}_0^J - \lambda H(q-q^c)\tilde{\rho}_0^J$$

Then the first moment of the jumping time is

$$E[\tau^J] = \int_0^\infty t\rho_{\tau^J}(t)\, dt = -\int_0^\infty t \int_{-\infty}^\infty \frac{d}{dt}\rho_0^J(q,t)\, dqdt$$
$$= \frac{D_0^2}{m}\left(\sqrt{m^2 + 2D_0^2\lambda} - m\right)^{-1}.$$

First Passage Time

$$\begin{cases} \partial_t \rho_1^F = r\partial_q \rho_1^F + \frac{D_1^2}{2}\partial_q^2 \rho_1^F \\ \rho_1^F(q,0) = \lim_{t\to\infty} \rho_1^J(q,t) \end{cases}$$

Note,

$$\lim_{t\to\infty} \rho_1^J(q,t) = \lambda H(q-q^c)\int_0^\infty \rho_0^J \, dt = \lambda H(q-q^c)\tilde{\rho}_0^J(q, s=0)$$

Similar as last slide,

$$E[\tau_{FP}] = \frac{q^\epsilon}{r} + \frac{D_0^2}{r(\sqrt{m^2+2D_0^2\lambda}-m)}.$$

$$E[\tau_{ET}] = E[\tau_{FP}+\tau^J] = \frac{q^\epsilon}{r} + C\lambda^{-1/2}$$

Know C exactly.

Convergence of Models

Question: are we justified in using (D2) to approximate (S2)?

Theorem

Let $(q_t^\lambda, \sigma_t^\lambda)$ *be the solution of SDE (S2) with initial conditions* (q_0, σ_0) *constant for every* λ *and let* (q_t, σ_t) *be the solution to SDE (D2) with the same initial condition* (q_0, σ_0)*. Then*

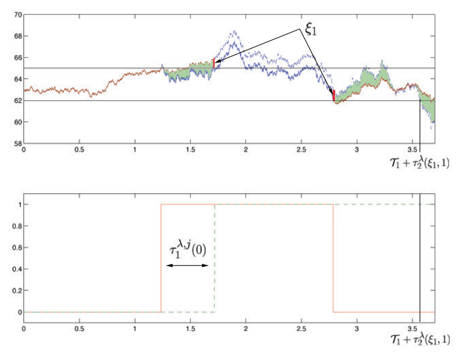
$$\lim_{\lambda\to\infty} E\left[|q_T^\lambda - q_T|^2\right] = 0.$$

Sketch of Proof

- First: processes don't jump too many times (only have finite number of jumps).
- Second: processes don't jump too few times (λ process is at most 1 jump behind)
- Third:

$$q_T^{\lambda} - q_T = \sum_{i=1}^{N} \xi_i + \zeta_i$$

Main Idea of the Proof



Two types of Error. 1) ξ from delay in jumping time. 2) ζ accruing "catch up" error.

Sketch of Proof

$$q_t^\lambda - q_t = \sum_{i=1}^{N} \xi_i + \zeta_i$$

- ξ_i are independent and identically distributed (IID) random variables.

$$E[|\xi_i|] = O(\lambda^{-1/2}) \quad E[|\xi_i|^2] = O(\lambda^{-1/2})$$

- ζ_i are not independent or identically distributed.

$$E[\zeta_i] = c_i E\left[\tau_i^\lambda \left(\sum_{j}^{i} \xi_j + \sum_{j=1}^{i-1} \zeta_j\right)\right]$$

$$E[|\zeta_i|^2] = O\left(\sum_{j=1}^{i-1}\sum_{k=1}^{j} d_k E[\xi_k]\right) = O(i^2\lambda^{-1/2})$$

Conclusions

- The two threshold models capture relevant observational statistics.
- While (S2) captures the statistics more accurately, the exact formulas are complex and in some cases, very hard to analyze.
- (D2) approximates (S2) with large λ.
- On going work:
  - Consider many columns of moisture interacting non-linearly.
  - The simplicity of (D2) and possibly (D1) will be key to studying multi-column models.
  - How do the relevant statistics change as the interactions change?