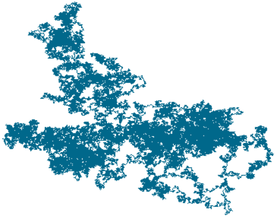
Noise-Induced Stabilization of Planar Flows[1]

David P. Herzog
Duke University



May 20th, 2014

[1]Joint work with J. Mattingly

Noise-induced stabilization

- **NIS** refers to the phenomenon that, under the addition of noise or randomness, a unstable deterministic dynamical system becomes stable.

NOISE-INDUCED STABILIZATION

- **NIS** refers to the phenomenon that, under the addition of noise or randomness, a unstable deterministic dynamical system becomes stable.
- M. Scheutzow (1993).

NOISE-INDUCED STABILIZATION

- **NIS** refers to the phenomenon that, under the addition of noise or randomness, a unstable deterministic dynamical system becomes stable.
- M. Scheutzow (1993).
- Here we discuss a family of explosive ODEs whose stochastic perturbations are quite stable. Specifically:

NOISE-INDUCED STABILIZATION

- **NIS** refers to the phenomenon that, under the addition of noise or randomness, a unstable deterministic dynamical system becomes stable.
- M. Scheutzow (1993).
- Here we discuss a family of explosive ODEs whose stochastic perturbations are quite stable. Specifically:
  - The stochastic dynamics generates a unique invariant measure μ.

NOISE-INDUCED STABILIZATION

- **NIS** refers to the phenomenon that, under the addition of noise or randomness, a unstable deterministic dynamical system becomes stable.
- M. Scheutzow (1993).
- Here we discuss a family of explosive ODEs whose stochastic perturbations are quite stable. Specifically:
  - The stochastic dynamics generates a unique invariant measure μ.
  - The stochastic dynamics approaches μ exponentially fast in time in a very strong sense.

NOISE-INDUCED STABILIZATION

- **NIS** refers to the phenomenon that, under the addition of noise or randomness, a unstable deterministic dynamical system becomes stable.
- M. Scheutzow (1993).
- Here we discuss a family of explosive ODEs whose stochastic perturbations are quite stable. Specifically:
  - The stochastic dynamics generates a unique invariant measure μ.
  - The stochastic dynamics approaches μ exponentially fast in time in a very strong sense.
  - μ decays polynomially at ∞.

Other consequences

- Arguments hinge on an intricate construction of a Lyapunov function.

OTHER CONSEQUENCES

- Arguments hinge on an intricate construction of a Lyapunov function.
- The construction procedure has a general framework.

OTHER CONSEQUENCES

- Arguments hinge on an intricate construction of a Lyapunov function.
- The construction procedure has a general framework.
- Produce “optimal” results.

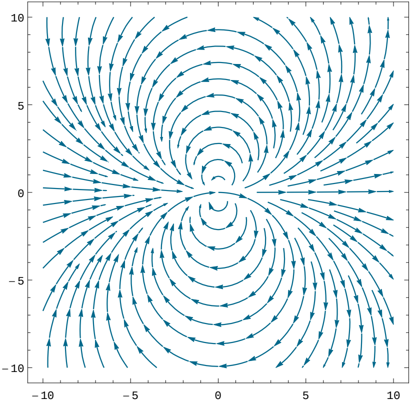
OTHER CONSEQUENCES

- Arguments hinge on an intricate construction of a Lyapunov function.
- The construction procedure has a general framework.
- Produce "optimal" results.
- Extends and improves a stream of previous results (J. Bec, M. Cencini, R. Hillerbrand, A. Athreya, T. Kolba, J. Mattingly, J. Wehr, K. Gawędzki, H.)

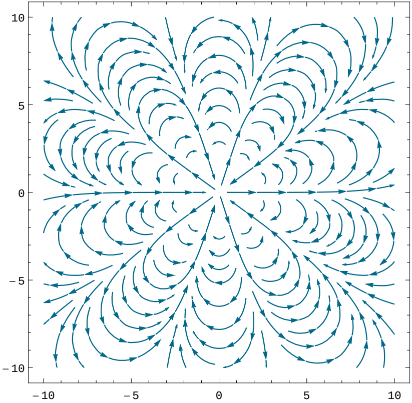
OTHER CONSEQUENCES

- Arguments hinge on an intricate construction of a Lyapunov function.
- The construction procedure has a general framework.
- Produce "optimal" results.
- Extends and improves a stream of previous results (J. Bec, M. Cencini, R. Hillerbrand, A. Athreya, T. Kolba, J. Mattingly, J. Wehr, K. Gawędzki, H.)
- Application and derivation of a generalized chain rule for the Itô calculus (G. Peskir '07).

THE DETERMINISTIC FAMILY

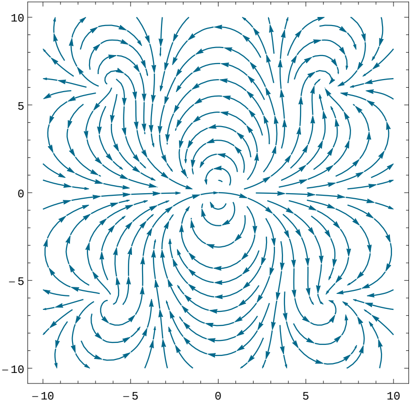


(a) $\dot{z}_t = z_t^2$

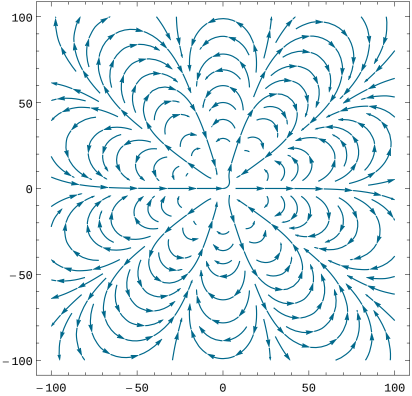


(b) $\dot{z}_t = z_t^6$

THE DETERMINISTIC FAMILY



(c) $\dot{z}_t = z_t^6 + 6000 z_t^2$



(d) $\dot{z}_t = z_t^6 + 6000 z_t^2$

THE STOCHASTIC PERTURBATIONS

Consider

$$dz_t = \left[z_t^n + F(z_t, \bar{z}_t)\right] dt + \sigma\, dB_t$$

where

- $n \geq 2$, $\sigma > 0$.
- $F(z, \bar{z})$ is a complex polynomial in $(z, \bar{z})$ with

$$F(z, \bar{z}) = \mathcal{O}(|z|^{n-1}) \text{ as } |z| \to \infty.$$

- $B_t = B_t^1 + iB_t^2$ where B_t^1 and B_t^2 are independent standard (real-valued) Brownian motions.

QUALITATIVE BEHAVIOR OF THE PERTURBATIONS

STABILIZATION

Theorem (H., Mattingly '13)

The process z_t is non-explosive and possesses a unique invariant measure μ. Moreover, there exist constants $r, C > 0$ such that

$$\|P_t(z, \cdot) - \mu(\cdot)\|_{TV} \leq Ce^{-rt}$$

for all $z \in \mathbb{C}$ and all $t \geq 0$.

BEHAVIOR OF THE INVARIANT MEASURE AT ∞

Theorem (H., Mattingly '13)

The integral

$$\int_{\mathbb{C}} (1+|z|)^{\delta} \mu(dz)$$

is finite if and only if $\delta < 2n-2$.

BEHAVIOR OF THE INVARIANT MEASURE AT ∞

Theorem (H., Mattingly '13)

The integral

$$\int_{\mathbb{C}} (1 + |z|)^{\delta} \mu(dz)$$

is finite if and only if $\delta < 2n - 2$.

Corollary

Recall that ρ *denotes the density of* μ. *Then there exists a positive constant* C *such that*

$$\liminf_{|z| \to \infty} \rho(z) \geq C|z|^{-2n}.$$

OVERVIEW IN THE GENERAL EQUATION

Preliminaries:

- Find $\varphi \in C^2(\mathbb{R}^2 : [0, \infty))$ satisfying

OVERVIEW IN THE GENERAL EQUATION

Preliminaries:

- Find $\varphi \in C^2(\mathbb{R}^2 : [0, \infty))$ satisfying
  (I) $\varphi(r, \theta) \to \infty$ as $r \to \infty$.

OVERVIEW IN THE GENERAL EQUATION

Preliminaries:

- Find $\varphi \in C^2(\mathbb{R}^2 : [0, \infty))$ satisfying
  - (I) $\varphi(r, \theta) \to \infty$ as $r \to \infty$.
  - (II) $(\mathcal{L}\varphi)(r, \theta) \to -\infty$ as fast as possible as $r \to \infty$. Should go to $-\infty$ slower than $-r^{2n-2}$.

OVERVIEW IN THE GENERAL EQUATION

Preliminaries:

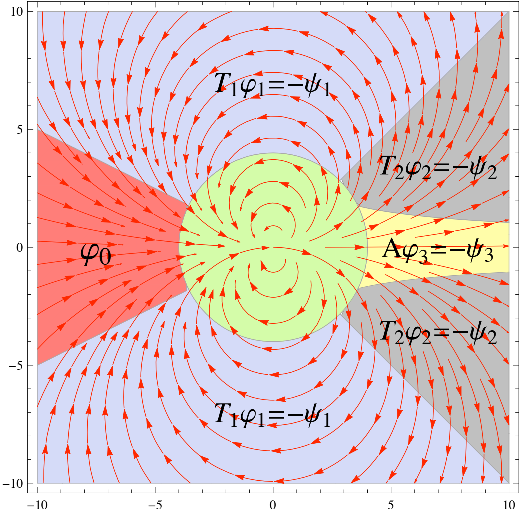
- Find $\varphi \in C^2(\mathbb{R}^2 : [0, \infty))$ satisfying
  - (I) $\varphi(r, \theta) \to \infty$ as $r \to \infty$.
  - (II) $(\mathcal{L}\varphi)(r, \theta) \to -\infty$ as fast as possible as $r \to \infty$. Should go to $-\infty$ slower than $-r^{2n-2}$.

OVERVIEW IN THE GENERAL EQUATION

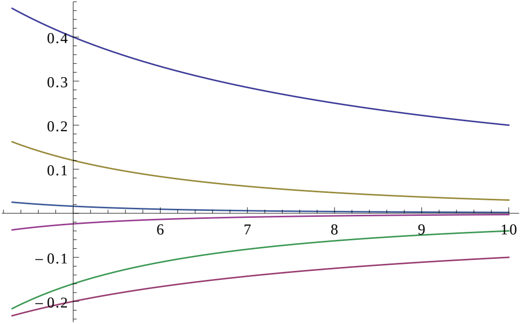
Preliminaries:

- Find $\varphi \in C^2(\mathbb{R}^2 : [0, \infty))$ satisfying
  - (I) $\varphi(r, \theta) \to \infty$ as $r \to \infty$.
  - (II) $(\mathcal{L}\varphi)(r, \theta) \to -\infty$ as fast as possible as $r \to \infty$. Should go to $-\infty$ slower than $-r^{2n-2}$.
  - (III) Then note also that $(\mathcal{L}\varphi)(r, \theta) \leq -C\varphi(r, \theta)^{1+\gamma} + D$ for all (r, θ) where $C, D, \gamma > 0$.

DEFINING φ



THE GENERAL CASE: FURTHER STRATIFICATION



Thanks for listening!

TANAKA'S FORMULA IN $d \geq 2$ DIMENSIONS

Let $\varphi \in C(\mathbb{R}^d : \mathbb{R})$ be such that

$$\varphi(x) = \begin{cases} \varphi_1(x) & \text{for } x^d \leq a(x^1, \ldots, x^{d-1}) \\ \varphi_2(x) & \text{for } x^d \geq a(x^1, \ldots, x^{d-1}) \end{cases}$$

where $\varphi_i \in C^2$ on its domain of definition. Suppose X_t has generator

$$\mathcal{L} = \sum_{j=1}^{d} b^j(x) \frac{\partial}{\partial x^j} + \frac{1}{2} \sum_{i,j=1}^{d} \sigma^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j}.$$

GENERALIZED DYNKIN FORMULA

$$\mathbb{E}_x\varphi(X_t) - \varphi(x)$$

$$= \mathbb{E}_x \int_0^t \frac{1}{2}\mathcal{L}\varphi(X_s^1, \ldots, (X_s^d)^+) + \frac{1}{2}\mathcal{L}\varphi(X_s^1, \ldots, (X_s^d)^-)\, ds + \textbf{FLUX}$$

GENERALIZED DYNKIN FORMULA

$$\mathbb{E}_x \varphi(X_t) - \varphi(x)$$

$$= \mathbb{E}_x \int_0^t \frac{1}{2}\mathcal{L}\varphi(X_s^1, \ldots, (X_s^d)^+) + \frac{1}{2}\mathcal{L}\varphi(X_s^1, \ldots, (X_s^d)^-)\, ds + \textbf{FLUX}$$

FLUX

$$= \frac{1}{2}\lim_{\epsilon \downarrow 0} \mathbb{E} \int_0^t \int_\Gamma \left(\frac{\partial \varphi_2}{\partial x^d}(y) - \frac{\partial \varphi_1}{\partial x^d}(y) \right) \rho_\epsilon(X_s - y)\sigma^{ij}(X_s)\nu^i \nu^j \, dS_\Gamma(y)\, ds.$$

TAILS OF THE INVARIANT MEASURE