

Pairing of Zeros and Critical Points for Random Polynomials

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- Flow lines for $-\nabla M(z)$ accumulate at zeros of p_N .

Pairing of Zeros and Crits for $SU(2)$ Polynomials



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- $\frac{d}{dz} p_N(z) = 0 \iff d \log |p_N(z)| = 0$

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Figure: Zeros – black discs. Crits – blue squares.

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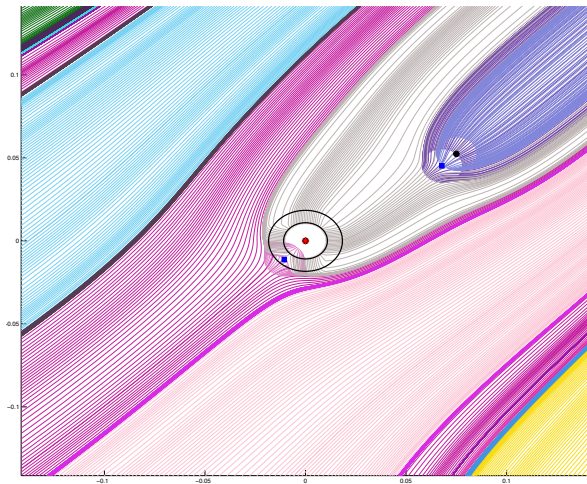
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Pairing Zeros with Critical Points for $SU(2)$ Polynomials

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$$\mathbb{P}(\exists! \epsilon - \text{paired w to } \xi) \geq 1 - K \cdot N^{-3/2+3\epsilon}.$$

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 - **Def.** $\xi \in S^2$ is distinguished in $HGE(\phi, N)$ if $d\phi(\xi) = 0$.

Proof Ingredients

- Local scaling limit of p_N is multiple of GAF:

$$p_N \left(\xi + \frac{u}{\sqrt{N}} \right) \cdot e^{\sqrt{N}\phi(\xi+u/\sqrt{N})} \rightarrow \sum_{j \geq 0} a_j \frac{u^j}{\sqrt{j!}}$$

in the sense of C^∞ convergence of covariance kernels.

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- Correlations — $\mathbb{E}[Z_{p_N} \otimes C_{p_N}] - \mathbb{E}[Z_{p_N}] \otimes \mathbb{E}[C_{p_N}]$

Nearest Neighbor Spacings Near $\xi = 0$

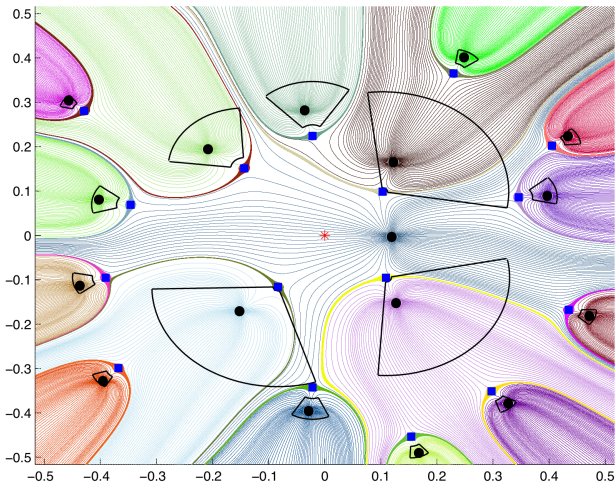


Figure: Zeros and critical points near $\xi = 0$. For every critical point the sector predicted to contain its paired zero is shown.

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We have, for each $\epsilon > 0$,

$$\mathbb{E}[\mathcal{X}_{A,N}] = \mathbb{E}[C_A] + O\left(\int_A |w|^{-1/4} dw \wedge d\bar{w}\right) + O(N^{-1/2+\epsilon}).$$