

Ornstein-Uhlenbeck processes for geophysical data analysis

Semere Habtemicael

Department of Mathematics
North Dakota State University

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Introduction

- In the study of relations between seismic dynamics and underlying physical processes, a major role is played by the question of how the *magnitude* rates evolve with time and if it is at all possible to have an a priori estimate of big jumps in the magnitude rate.
- Recent research shows a renewed interest in describing *critical phenomena* using several modeling techniques.
- For example, Ising models, phase transition, fitting the data with exponential sequence, or so-called scale invariance property.

- In a recent work scale invariant functions and stochastic Lévy models are applied to geophysical data and it is shown that a pattern arises from the scale invariance property and Lévy flight models that may be used to estimate parameters related to some major event- major earthquakes.
- Past literature also used the generalized *Omori law* and *ETAS* model (Epidemic-Type aftershock sequence) for quantitative statistic modeling of seismic regime.

- However, it appears that there is a major drawback with the procedure described in the previous works.
- Firstly, it is now well accepted that an earthquake model is *not* a deterministic one and hence the deterministic models such as scale-invariance technique may not be a suitable model.
- For the existing stochastic models there is no concrete theory that models the data point and estimate the earthquake data from a separate estimation depending on the model parameters.

For a particular earthquake prone region if the time series of data points (magnitude) of an earthquake are joined with lines then the following properties are clear for any earthquake data:

- ① Magnitude is a non-negative stationary stochastic process.
- ② For any finite interval of time there are only finite number of jumps.
- ③ The sample path of magnitude of earthquake consists of upward jumps (significant earthquake) and gradual decrease (aftershocks).

- In our work we show that there is a non-Gaussian Ornstein-Uhlenbeck (OU) process that satisfies all the above properties and hence that process is a possible candidate for modeling the earthquake data.
- Non-Gaussian processes of OU type have considerable potential as building-blocks for different stochastic models of observational time series from a variety of fields.
- They offer the possibility of capturing important distributional deviations from Gaussianity and provide flexibility in modeling dependence structure.
- Such models are widely used in finance for modeling the market *volatility*.

- When Lévy processes are used as driving noise it is possible to construct a large family of *mean-reverting* jump processes with linear dynamics on which various properties such as positiveness or the choice of marginal distribution, can be imposed.
- We consider continuous time stationary and non-negative processes which are defined by the following stochastic differential equation

$$dM_t = -\lambda M_t dt + dZ_{\lambda t}, \quad M_0 > 0, \quad (1)$$

where the process Z_t is a subordinator- that is, it is a Lévy process with no Gaussian component and positive increments.

- The solution of (1) can be explicitly written as

$$M_t = \exp(-\lambda t)M_0 + \int_0^t \exp(-\lambda(t-s)) dZ_{\lambda s}, \quad (2)$$

which can also be written as $M_t = e^{-\lambda t}M_0 + e^{-\lambda t} \int_0^{\lambda t} e^s dZ_s$.

- As Z is an increasing process and $M_0 > 0$, the process (M_t) is strictly positive and it is bounded from below by the deterministic function $\exp(-\lambda t)M_0$.
- Since the process Z_t has positive increments and no drift, (M_t) moves up entirely by jumps and then tails off exponentially.

Characteristic function

- Suppose $\kappa(\theta) = \log E[e^{\theta Z_1}]$, is the cumulant transform for Z_1 .
- Estimation of a major event in a future time depends on the analysis of the conditional distribution of the magnitude process M_T given the information up to a certain time t_0 .

Characteristic function

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- Estimation of a major event in a future time depends on the analysis of the conditional distribution of the magnitude process M_T given the information up to a certain time t_0 .
- In practice t_0 is taken to be the time till which the magnitude data for the geophysical process is available and T is taken to be some reasonable time after t_0 on which the estimation of future *major event* is computed.
- The characteristic function for M_T given the information up to time $t_0 \leq T$ is given by $E[\exp(iuM_T)|\mathcal{F}_{t_0}]$, where $u \in \mathbb{R}$. The conditional distribution of M_T given the information up to time $t_0 \leq T$ will be denoted by $M_{T|t_0}$.

Theorem

Let Z be a subordinate with cumulant transform κ and let $f : \mathbb{R}^+ \rightarrow \mathbb{C}$ be a complex-valued, left continuous such that $\Re(f) \leq 0$ then

$$E \left[\exp \left(\int_0^t f(s) dz_{\lambda s} \right) \right] = \exp \left(\int_0^t \kappa(f(s)) ds \right) \quad (3)$$

Theorem

Consider the OU-process given by (1). Then the characteristic function $\phi(u) = E[\exp(iuM_T)|\mathcal{F}_{t_0}]$ of $M_T|_{t_0}$ is given by

$$\phi(u) = \exp \left[iuM_{t_0}e^{-\lambda T} + \lambda \int_{t_0}^T \kappa \left(iue^{-\lambda(T-s)} \right) ds \right], \quad (4)$$

where $\kappa(\cdot)$ is the cumulant transform for Z_1 .

- It is well known that variance gamma distribution is self-decomposable. If the stationary distribution of (M_t) is given by $\Gamma(\nu, \alpha)$ law then

$$\kappa(\theta) = \frac{\nu\theta}{\alpha - \theta}. \quad (5)$$

- For $\Gamma(\nu, \alpha)$ distribution (living on \mathbb{R}^+) the probability density and Lévy density are $p(x) = \frac{\alpha^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\alpha x}$, and $u(x) = \nu x^{-1} e^{-\alpha x}$ respectively.
- For the corresponding BDLP Z has Lévy density (living on \mathbb{R}^+)

$$w(x) = \alpha\nu e^{-\alpha x}.$$

From this it can be easily derived that the BDLP for the $\Gamma(\nu, \alpha)$ -OU process is a *compound Poisson process*

$$Z_t = \sum_{n=1}^{N_t} x_n,$$

where $(N_t)_{t \geq 0}$ is a Poisson process with $E(N_t) = \nu t$ and $\{x_n, n = 1, 2, \dots\}$ is an independent and identically distributed sequence with each x_n has $\Gamma(1, \alpha)$ (that is $Exp(\alpha)$) law.

We now compute the characteristic function for M when it is Gamma-OU process. Using (5) we obtain

$$\begin{aligned} & \int_{t_0}^T \kappa \left(iue^{-\lambda(T-s)} \right) ds \\ &= -\frac{\nu}{2\lambda} [2\lambda(T-t_0) + \log(\alpha^2 + u^2) - \log(\alpha^2 e^{2\lambda(T-t_0)} + u^2)] \\ &+ 2i \left[\arctan\left(\frac{\alpha}{u}\right) - \arctan\left(\frac{\alpha e^{\lambda(T-t_0)}}{u}\right) \right]. \end{aligned}$$

Thus for Gamma-OU process

$$\begin{aligned} \phi(u) = & \exp\left[-\frac{\nu}{2} \left[2\lambda(T-t_0) + \log(\alpha^2 + u^2) - \log(\alpha^2 e^{2\lambda(T-t_0)} + u^2)\right]\right] \\ & + i \left[uM_{t_0} e^{-\lambda T} - \nu \arctan\left(\frac{\alpha}{u}\right) + \nu \arctan\left(\frac{\alpha e^{\lambda(T-t_0)}}{u}\right) \right]. \end{aligned} \tag{6}$$

$$|\phi(u)| = \left(\frac{\alpha^2 + u^2 e^{-2\lambda(T-t_0)}}{\alpha^2 + u^2} \right)^{\frac{\nu}{2}} \quad (7)$$

The theoretical $|\phi(u)|$ for Gamma-OU processes are fitted with the absolute value of the characteristic function computed from the observed data to estimate the appropriate values of λ , ν , and α .

First-passage time

- If b is a threshold of the magnitude above which earthquakes can be attributed as “major” or “devastating” for a certain geographical region, then it is important to understand the exit time given by

$$\tau_b = \inf\{t > 0 : M_t \geq b\},$$

for some given value $b > M_0$.

- The distribution of τ_b is well understood for the Gaussian OU processes from the pioneering work. However, this problem is significantly complicated for non-Gaussian OU processes.

Theorem

Let M be a $\Gamma(\nu, \alpha)$ -OU process given by (1). Then the expected value of τ_b satisfies the following relation

$$E(\tau_b) \geq \frac{1}{\lambda} \int_0^\alpha \frac{(e^{ub} - e^{uM_0})}{u} \left(\frac{\alpha - u}{\alpha} \right)^\nu du. \quad (8)$$

Regression

- Based on the data of the magnitude process $(M_t)_{0 \leq t \leq t_0}$ it is possible to compute the characteristic function $\phi_O(u) = E[\exp(iuM_T) | \mathcal{F}_{t_0}]$, where the suffix 'O' of ϕ stands for "observed" characteristic function based on the data set up to time t_0 .
- We numerically compute the probability distributions and then use *Fast Fourier Transform* (FFT) to compute $\phi_O(u)$.

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- We numerically compute the probability distributions and then use *Fast Fourier Transform* (FFT) to compute $\phi_O(u)$.
- For the Gamma-OU model, we fit theoretical $|\phi(u)|$ to the observed $|\phi_O(u)|$ using non-linear regression.
- The fitting minimizes *root-mean square error* (RMSE).
- This gives the best fit values of the parameters α , ν and λ .

- We find for each specific region, p -value is much less than 0.05 for all the parameters which shows that *all* the three parameters (α , ν and λ) are very significant for the model.
- Finally (8) is used to compute the lower bound of the expected time of a *major* earthquake.
- At that time the awareness level should be raised significantly in anticipation of a *major* earthquake.

- We note that (8) does *not* give a lower bound for τ_b . Instead it gives the lower bound for $E(\tau_b)$ which is a statistical property of τ_b .
- We acknowledge the fact that in some exceptional cases it is still possible to get a *major* earthquake even before the lower bound time of $E(\tau_b)$.

- We note that (8) does *not* give a lower bound for τ_b . Instead it gives the lower bound for $E(\tau_b)$ which is a statistical property of τ_b .
- We acknowledge the fact that in some exceptional cases it is still possible to get a *major* earthquake even before the lower bound time of $E(\tau_b)$.
- However, statistically those incidences are very less probable.
- Partial compensation of the lower bound of $E(\tau_b)$ can be attained by little flexibility in the values of b in the definition of *major* earthquake. For our calculation b is chosen from a range 6.2 to 6.7.

- The geophysical data was obtained from U.S. Geological Survey (USGS) from January 1, 1973 to November 9, 2010.
- This data contains information about the date, longitude, latitude, and magnitude of each recorded earthquake in the region.
- This study will be characterized by the geographical region in a neighborhood of location of the major earthquake.

- The choice of this region should be done carefully. This area can not be too small because in that case there will be not enough data to run the regression analysis.
- The area can not be too big because in that case the magnitude data set is distorted due to *noise* from unrelated events.
- The data is obtained using a *square* centered at the coordinates of the major event. The sides of the square are usually chosen as $\pm 0.1^\circ$ – 0.5° in latitude and $\pm 0.2^\circ$ – 0.5° in longitude. A segment 0.1° of latitude at the equator is ≈ 6.9 miles ≈ 11.11 km in length.
- Major earthquakes are earthquakes having magnitude greater than 7.

The following Figures (Figure 1 to Figure 5) present the fitting of absolute value of the characteristic function. In each of the figures, circles denote values of $|\phi_O(u)|$ which are computed from actual data points and the solid line is the regression fitting.

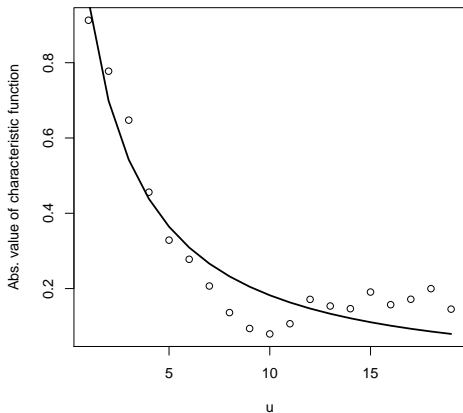


Figure: Fitting for Latitude $40.37^\circ \pm 0.1^\circ$ and Longitude $-124.32^\circ \pm 0.2^\circ$.

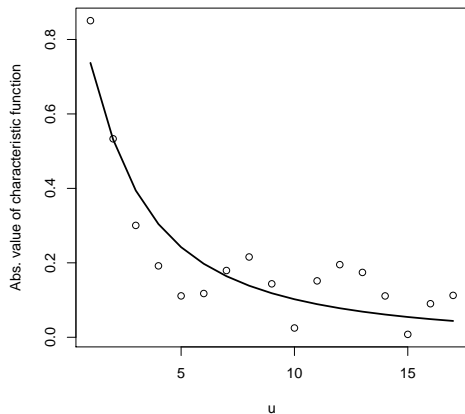


Figure: Fitting for Latitude $-23.34^\circ \pm 0.5^\circ$ and Longitude $-70.30^\circ \pm 0.5^\circ$.

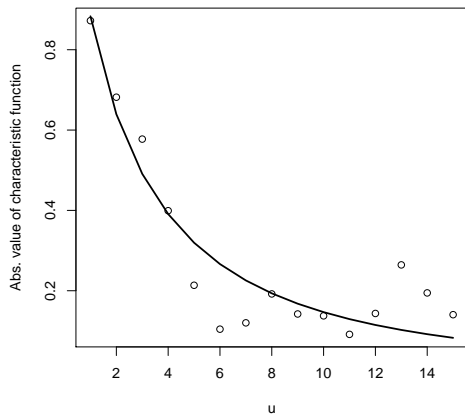


Figure: Fitting for Latitude $63.52^\circ \pm 0.17^\circ$ and Longitude $-147.44^\circ \pm 0.34^\circ$.

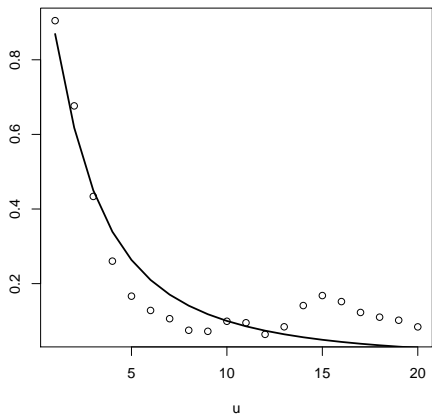


Figure: Fitting for Latitude $-17.66^\circ \pm 0.03^\circ$ and Longitude $-178.76^\circ \pm 0.06^\circ$.

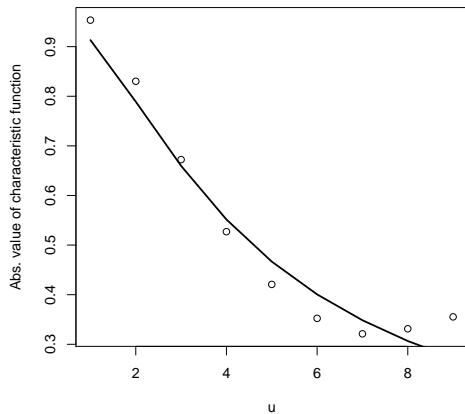


Figure: Fitting for Latitude $-19.93^\circ \pm 0.05^\circ$ and Longitude $-178.18^\circ \pm 0.1^\circ$.

The table below shows the estimate of the three parameters using regression analysis.

Table: Estimation from the regression model.

Region	M_0	t_0	α	ν	λ	RMSE
1	3.9	12/20/1991	2.4113	1.2045	2.3879	0.06346
2	4.6	06/01/1995	2.0007	0.4244	0.9951	0.05913
3	4.2	11/03/2002	2.8355	2.3713	0.99289	0.06875
4	4.1	07/15/2004	2.7123	2.7231	0.99052	0.04582
5	4.7	08/25/2005	3.2601	1.6960	0.99552	0.03451

In the following table t_r is the estimated date on which alert should be raised about possible earthquake in near future, and t_m is the actual known *major* earthquake date.

Table: Estimation from the regression model.

Latitude	Longitude	$t = 0$	t_r	t_m
$40.37^\circ \pm 0.1^\circ$	$-124.32^\circ \pm 0.2^\circ$	09/30/1973	6572	6782
$-23.34^\circ \pm 0.5^\circ$	$-70.30^\circ \pm 0.5^\circ$	07/13/1973	7759	8052
$63.52^\circ \pm 0.17^\circ$	$-147.44^\circ \pm 0.34^\circ$	03/26/1976	10632	10753
$-17.66^\circ \pm 0.03^\circ$	$-178.76^\circ \pm 0.06^\circ$	01/17/1973	11394	11501
$-19.93^\circ \pm 0.05^\circ$	$-178.18^\circ \pm 0.1^\circ$	04/04/1976	10454	10865

Conclusion

- ① In this paper we used the Gamma- OU process to model magnitude process of earthquake data.
- ② The estimated time is typically preceding the time of the major event. This methodology can be used in real time by looking at the minor earthquakes up to a certain data to estimate a major earthquake in the future.
- ③ The three parameter of Gamma-OU model is found using the regression analysis.
- ④ Additional model is in progress on this aspect in order to improve the present model.

Thank You!

Questions?...