

A New Method to Generate Uniform Random Variates on the Unit Spheric Shell in \mathbb{R}^d

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Isotropic Densities

An isotropic density $\psi(x)$ is defined as one that is invariant to rotations:

$$\psi(x) = \phi(\|x\|) \text{ for some function } \phi,$$

e.g. the multivariate normal and the multivariate Student's t .

Isotropic densities can be written as the product of another two random variables, one of them uniformly distributed on the unit spheric shell.

Uniform Random Variates on the Spheric Shell

Different algorithms to generate uniform random variates on the spheric shell have been proposed in the literature, most of them relying on the generation of a particular isotropic random variate which is then normalized (see [Devroye(1986)] for a detailed discussion).

In this paper we discuss alternative approaches that generate directly from the distribution rather than indirectly through normalization.

General Methodology

$$B_d(r) = \{x \in \mathbb{R}^d \mid \|x\| = r\} \text{ and } \Xi = (\xi_1, \dots, \xi_d) \sim \text{Unif}[B_d(1)]$$

Our algorithm relies on decomposing the uniform density of Ξ as

$$f(\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_d) = f(\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_d \mid \xi_i) f(\xi_i),$$

where the first term of the decomposition reduces (after an appropriate rescaling) to $\text{Unif}[B_{d-1}(1)]$.

Applying this idea recursively only requires us to generate from the marginal distribution of a single component of Ξ for every value of d .

Background Propositions I

Proposition

For all fixed $d \geq 2$ the marginal density of ξ_i is

$$h(\xi_i | d) = \begin{cases} \frac{\omega_{d-1}}{\omega_d} (1 - \xi_i^2)^{\frac{d-3}{2}} & -1 \leq \xi_i \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

where ω_d is the area of the unit spheric shell on \mathbb{R}^d .

Proposition

Let $\xi_{-i} = (\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_d)$. The conditional density $f(\xi_{-i} | \xi_i)$ is the uniform on $B_{d-1}(\sqrt{1 - \xi_i^2})$.

Background Propositions II

Proposition

If $\Xi \sim \text{Uniform}[B_d(r)]$ and $\Psi = t \cdot \Xi$, then $\Psi \sim \text{Uniform}[B_d(tr)]$.

Algorithm

Proposition

The algorithm defined by the steps

- 1 Initialize $r_0 = 1$
- 2 For all $i = 1, \dots, d - 2$:
 - 1 Generate $\xi_i^* \sim h(\xi_i^* \mid d - i + 1)$, where h was defined in Equation (1).
 - 2 Calculate $\xi_i = r_i \cdot \xi_i^*$
 - 3 Set $r_i = \sqrt{r_{i-1}^2 - \xi_i^2}$
- 3 Set $\xi_1 = r_{d-2}\xi_1^*$ and $\xi_2 = r_{d-2}\xi_2^*$, where $(\xi_1^*, \xi_2^*) \sim \text{Uniform}[B_0(2)]$

generates uniform random variates on $B_d(1)$.

Inversion

In order to completely specify the previous algorithm it is necessary to establish a method to generate the random variates mentioned in step 2.1.

The simplest choice is to use the inversion technique. It is straightforward to find the distribution function corresponding to the density in (1).

Unfortunately when $d \geq 4$ the distributions are not analytically invertible.

Transformation I

Any algorithm for the generation of beta or gamma random variables can be used to produce random variables distributed according to the density in (1), since:

Proposition

Let X and Z be two random variables such that $X = 2Z - 1$. Then X has density (1) if and only if $Z \sim \text{Beta} \left(\frac{d-1}{2}, \frac{d-1}{2} \right)$.

Proposition

If $X, Y \sim \text{Gamma} \left(\frac{d-1}{2}, a \right)$ with $d > 1$ and $a > 0$, then the density of $Z = \frac{X-Y}{X+Y}$ reduces to that in Equation (1).

Transformation II

Note that using this property together with our basic algorithm is equivalent to the method described by [Hicks and Wheeling(1959)].

Finally, also note that this proposition can be used to recursively compute the area of a sphere of radius 1.

Acceptance-Rejection Methods I

Of particular interest are algorithms for generating Beta random variates that take advantage of the particular features of the distribution in (1), i.e. its symmetry and the form of its parameter (which is an integer multiple of $1/2$).

The density

$$g(x | s, c) = \frac{(1 + s^c)^{\frac{1}{c}}}{s} \left[1 + \left(\frac{x}{s} \right)^c \right]^{-\left(1 + \frac{1}{c}\right)}, \quad 0 \leq x \leq 1, \quad s > 0 \quad (2)$$

which belongs to the Burr III family [Burr(1942), Burr and Cislak(1968), Burr(1973)] is proposed as the blanketing function.

Acceptance-Rejection Methods II

Note that this choice is similar to that of [Cheng(1978)].

Choosing appropriate parameters the rejection is at about or below 9%.

The main drawback of this method is that the evaluation of the rejection condition is time consuming.

Piecewise linear or quadratic quick-acceptance and quick-rejection steps are suggested.

Ratio-of-Uniforms Method I

The ratio-of-uniforms method [Kinderman and Monahan(1977)] is an alternative acceptance-rejection method to generate from density (1). In this case, a pair (U, V) must be generated uniformly in the region

$$\Omega = \left\{ (u, v) \mid v^2 \leq u^2 \left(1 - u^{\frac{2}{r}} \right), r = \frac{d-3}{2} \right\}$$

and then the quotient $Z = \frac{U}{V}$ follows the density in (1). The pair (U, V) is obtained by rejection sampling from the square

$$M = \left\{ (u, v) \mid 0 \leq u \leq 1, |v| \leq \sqrt{\frac{r^r}{(1+r)^{1+r}}}, r = \frac{d-3}{2} \right\}$$

Ratio-of-Uniforms Method II

which completely encloses Ω .

Despite the fact that the rejection is greater than 35%, the evaluation of the rejection requires less operations. Also, due to the shape of the region, the squeeze function

$$f^*(x | d) = \begin{cases} |v| \leq \alpha_0 u & u \leq \left(\frac{r}{1+r}\right)^{r/2} \\ |v| \leq \alpha_1(1-u) & u > \left(\frac{r}{1+r}\right)^{r/2} \end{cases} \quad (3)$$

with $\alpha_0 = \sqrt{1+r}$ and $\alpha_1 = \frac{r^{r/2}}{\sqrt{1+r}[(1+r)^{r/2} - r^{r/2}]}$ can be used as a simple yet powerful quick-rejection step, avoiding the calculation of roots exactly 50% of the time.

Conclusion

We provide different algorithms to generate uniform random variates on the unit spheric shell.





All of the alternatives presented here are based on recursive generation of random variates from the marginal distribution, being of particular interest the ratio-of-uniforms method presented in the last section.

Extensions




The ideas presented here can be the basis to build a multidimensional extension of the Box-Muller method for isotropic densities (not only for the multivariate Normal but also for the multivariate Student's t distribution).

Also, our algorithm can be used to extend the algorithms based on the properties of spacings to odd dimensions.

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