The Basel problem revisited: a quantum probabilistic proof

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The Basel Problem

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots = \frac{\pi^2}{6}$$

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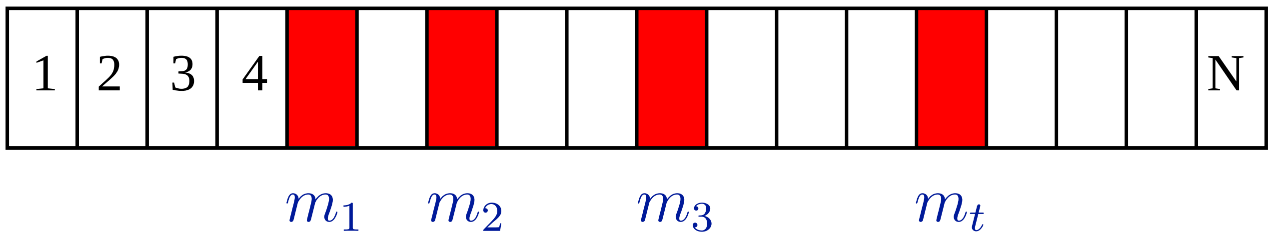
$\frac{4\pi^2}{25(3-\varphi)}$, where φ is the golden ratio

Counting problem

Problem: Given a population with N items with some colored red, how many red ones are there?



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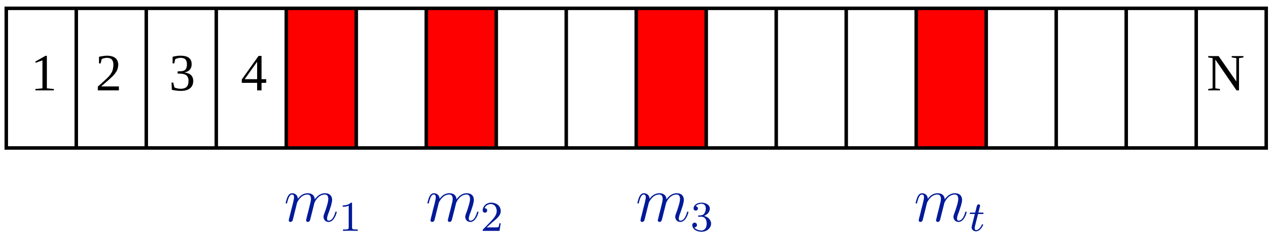

Oracle formulation: For $x = 1, 2, \ldots, N$,

$$f(x) = \begin{cases} 1, & \text{if } x = m_1, m_2, m_3, \ldots, m_t; \\ 0, & \text{otherwise.} \end{cases}$$

Find the number t by evaluations of f (oracle calls).

Classical proportion estimation method

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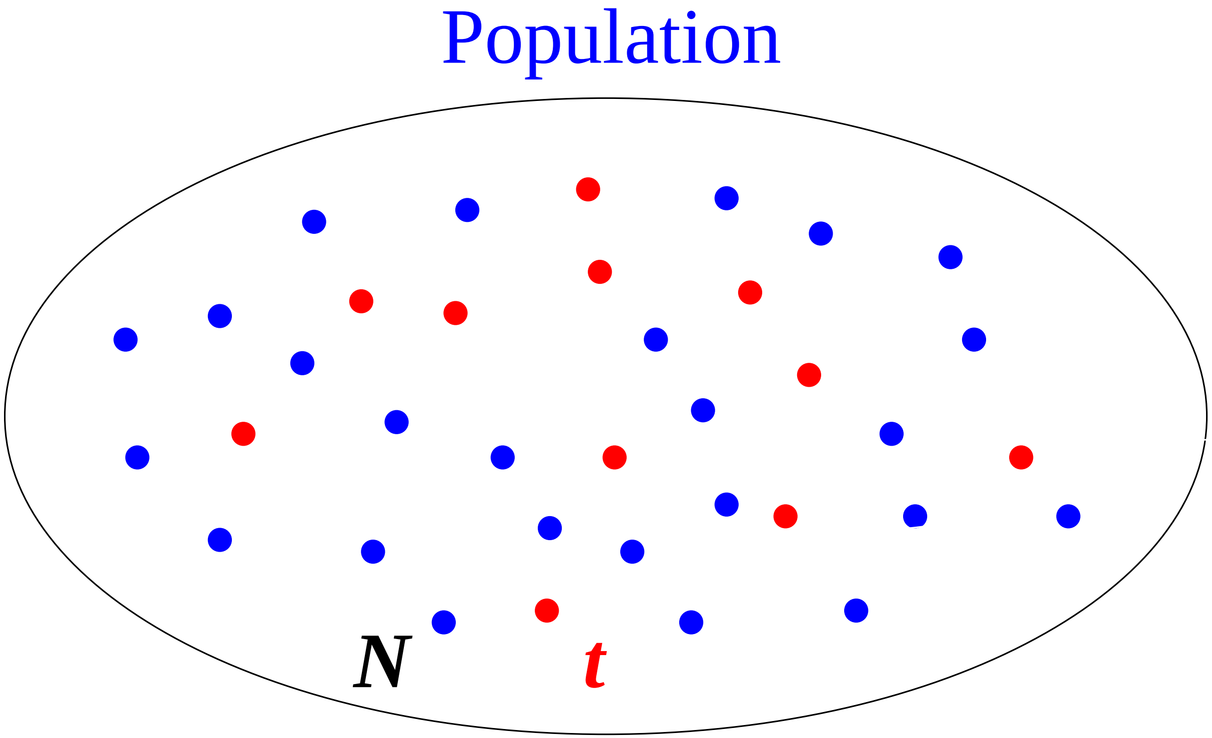
Finding $t \Leftrightarrow$ Finding $p = \dfrac{t}{N}$



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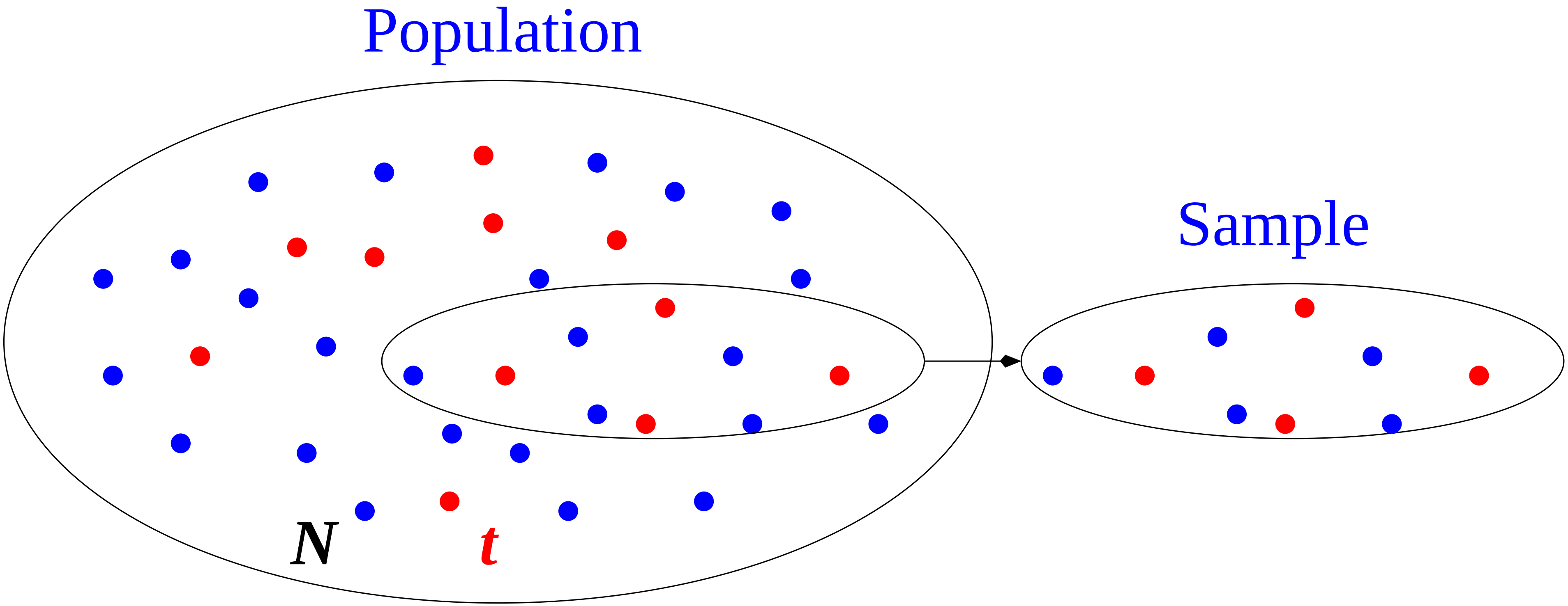

Draw a sample. Use the sample proportion $\hat{p}$ to estimate p.

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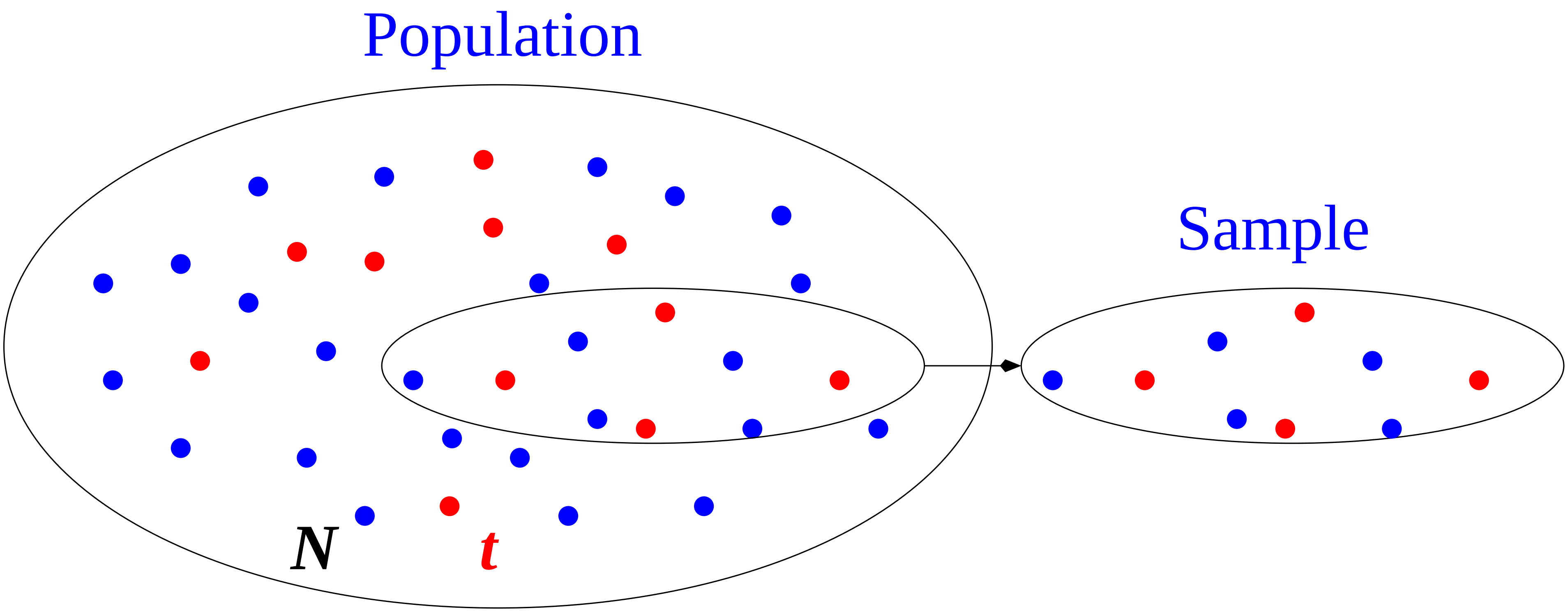

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How would a quantum computer do this?

Finding a needle in a haystack

Problem: Given an unsorted database with N items with one colored red, how to find the red one?

1	2	3	4						m								N

Oracle formulation:

$$f(x) = \begin{cases} 1 & \text{if } x = m \\ 0 & \text{if } x \neq m \end{cases}, \quad x = 1, 2, \ldots, N.$$

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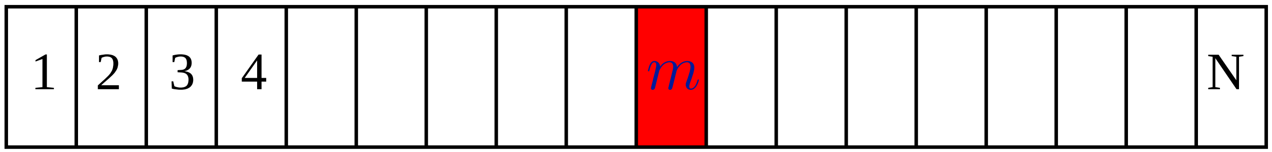

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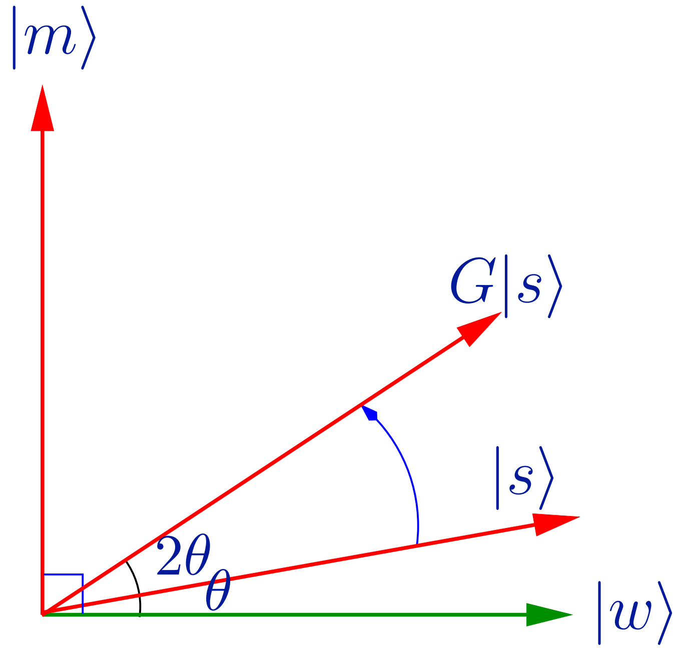
Classical search (brute force): $O(N)$ oracle calls

Quantum search: $O(\sqrt{N})$ oracle calls

Grover's algorithm

1. Prepare the initial state vector $|s\rangle = \frac{1}{\sqrt{N}} \sum_{x=1}^{N} |x\rangle$.
2. Apply **amplitude amplification operator** $G = \mathcal{I}_s \mathcal{I}_f$ on $|s\rangle$ for approximately $\frac{\pi}{4}\sqrt{N}$ times.
   - Selective sign flipping operator
     $$\mathcal{I}_f|x\rangle = (-1)^{f(x)}|x\rangle = \begin{cases} -|x\rangle, & \text{if } f(x) = 1 \\ |x\rangle, & \text{if } f(x) = 0 \end{cases}$$
   - Inversion around the average operator
     $$\mathcal{I}_s = 2|s\rangle\langle s| - \mathbb{I}$$
3. Measure.

Amplitude Amplification Operator



$$f(x) = 1 \leftrightarrow |m\rangle$$

$$f(x) = 0 \leftrightarrow |w\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq m} |x\rangle$$

Under basis $\{|w\rangle, |m\rangle\}$

$$G = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

$$\sin\theta = \langle s|m\rangle = \sqrt{\frac{t}{N}} = \sqrt{p}$$

G has eigenvectors of $|\psi_+\rangle$ and $|\psi_-\rangle$ with eigenvalues $e^{2\theta i}$ and $e^{-2\theta i}$, respectively.

QFT and inverse QFT

For $x, y \in \{0, 1, 2, \ldots, M-1\}$,

$$QFT : |x\rangle \to \frac{1}{\sqrt{M}} \sum_{y=0}^{M-1} e^{2\pi i \frac{x}{M} y} |y\rangle,$$

$$QFT^{-1} : |x\rangle \to \frac{1}{\sqrt{M}} \sum_{y=0}^{M-1} e^{-2\pi i \frac{x}{M} y} |y\rangle.$$

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Inverse Fourier Transform Formula

$$QFT^{-1}\left(QFT|x\rangle\right) = QFT^{-1}\left(\frac{1}{\sqrt{M}} \sum_{y=0}^{M-1} e^{2\pi i \frac{x}{M} y} |y\rangle\right) = |x\rangle$$

Frequency estimation via QFT

$$QFT^{-1}\left(\frac{1}{\sqrt{M}}\sum_{y=0}^{M-1} e^{2\pi i \frac{x}{M} y}|y\rangle\right) = |x\rangle$$

Discrete $\frac{x}{M} \Rightarrow$ Continuous $\omega \Downarrow$

$$QFT^{-1}\left(\frac{1}{\sqrt{M}}\sum_{y=0}^{M-1} e^{2\pi i \omega y}|y\rangle\right) \triangleq |\tilde{x}\rangle$$

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- $\widetilde{\omega} = \frac{x}{M}$ is a plausible estimator of ω.

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Theorem 1. *The probability of $\widetilde{\omega}$ being within $\frac{1}{M}$ of ω is at least $\frac{8}{\pi^2}$.*

Road map for quantum counting

Goal: $p = \frac{t}{N} = \sin^2 \theta = \sin^2 \pi\omega$

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$$\Downarrow(G^y)$$

$$G^y|s\rangle = \frac{e^{i\pi\omega}}{\sqrt{2}}e^{2\pi i\omega y}|\psi_+\rangle + \frac{e^{-i\pi\omega}}{\sqrt{2}}e^{-2\pi i\omega y}|\psi_-\rangle$$

$\Downarrow$(A procedure to manufacture superpositions)

$$\frac{1}{\sqrt{M}}\sum e^{2\pi i\omega y}|y\rangle$$

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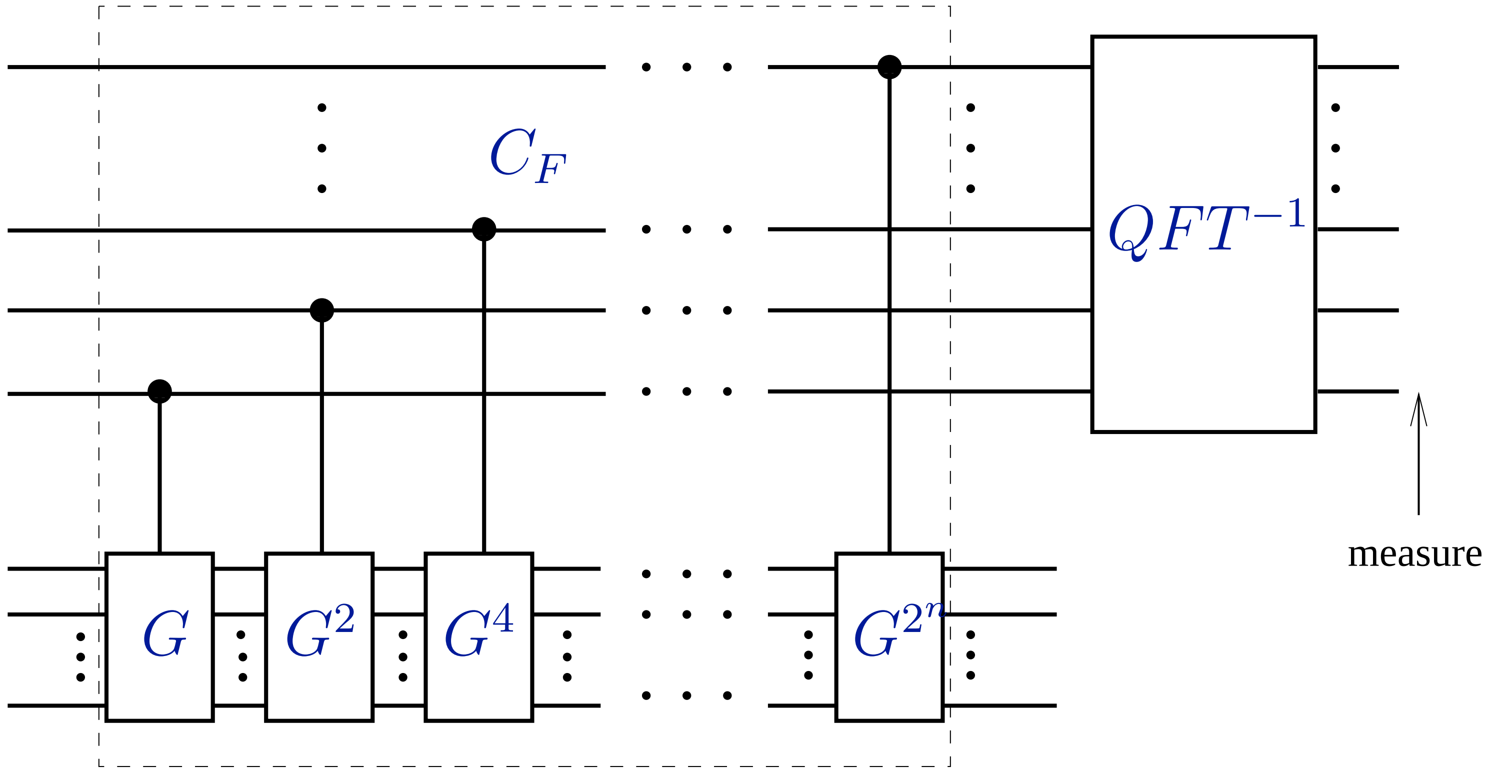
Output: $\tilde{p} = \sin^2(\pi\widetilde{\omega}) \approx \sin^2\pi\omega = p$

Algorithm

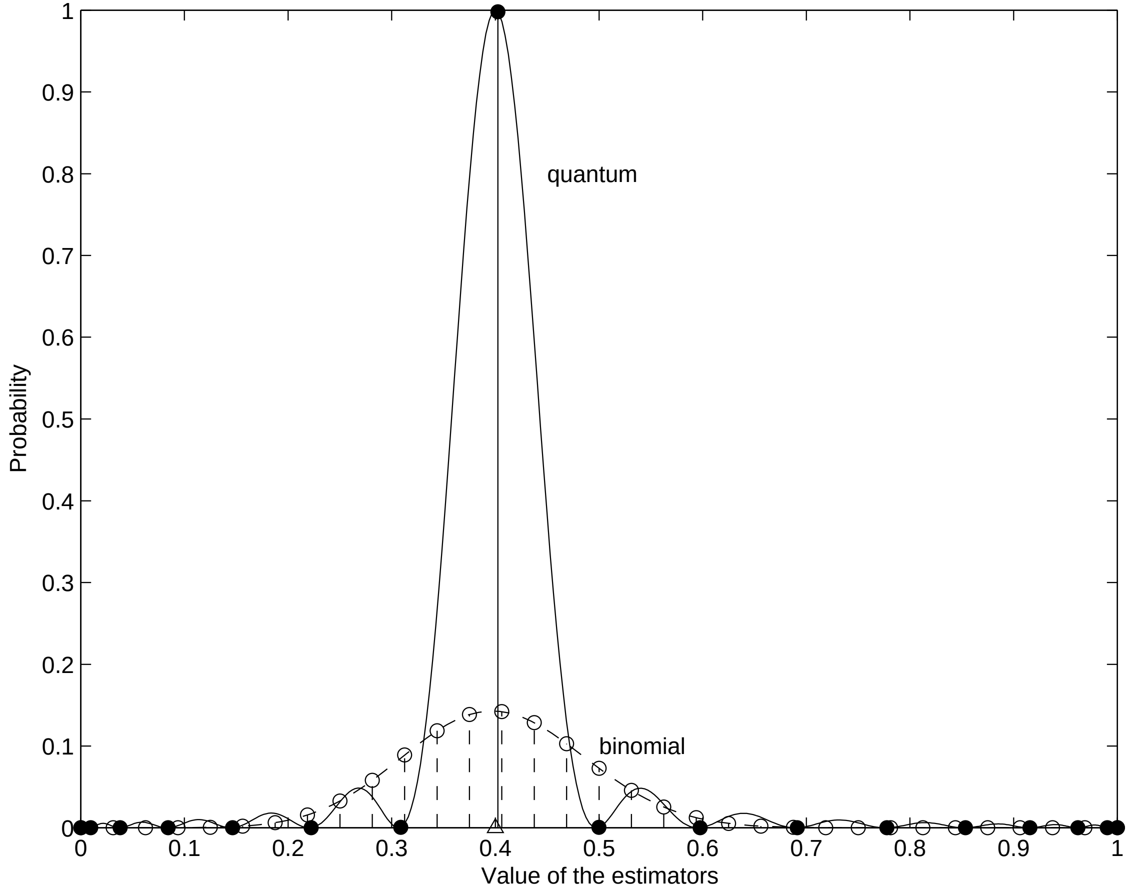
1. Prepare two registers in initial state $|\psi_0\rangle = \frac{1}{\sqrt{M}} \sum_{y=0}^{M-1} |y\rangle|s\rangle$.
2. Apply C_F on $|\psi_0\rangle$, which implements $|y\rangle|s\rangle \to |y\rangle G^y|s\rangle$.
3. Apply QFT^{-1} on the first register.
4. Measure the first register to obtain $|x\rangle$ and output $\tilde{p} = \sin^2(\pi \frac{x}{M})$.



Probability distribution of $\tilde{p}$

$$P\left[\tilde{p} = \sin^2\left(\frac{\pi x}{M}\right)\right] = \frac{\sin^2 M\theta}{M^2 \sin^2\left(\theta - \frac{\pi x}{M}\right)},$$ where $\theta = \sin^{-1}\sqrt{p}$.

$p = 0.4$: $N = 100$, $t = 40$, and $M = 32$.



Convergence in probability

Theorem 2. *As $M \to \infty$, $\tilde{p} \to p$ in probability.*

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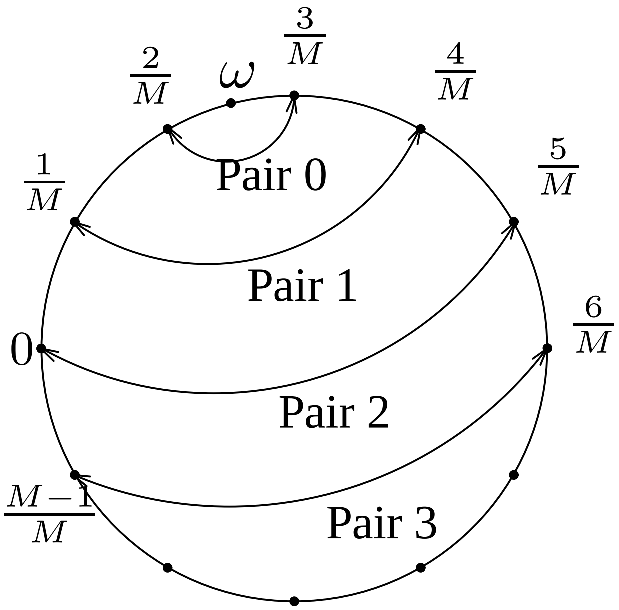
Proof.



$$\lim_{M\to\infty} P(|\tilde{p}-p|<\epsilon)$$
$$\geq \frac{8}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$
$$= \frac{8}{\pi^2}\cdot\frac{\pi^2}{8} = 1.$$

□

Back to the Basel problem

$$\sum_{x=0}^{M-1} P\left[\tilde{p} = \sin^2\left(\frac{\pi x}{M}\right)\right] = \sum_{x=0}^{M-1} \frac{\sin^2 M\theta}{M^2 \sin^2\left(\theta - \frac{\pi x}{M}\right)} = 1$$

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- Set $\theta = \dfrac{\pi}{2M}$:

$$\frac{2}{M^2}\left(\frac{1}{\sin^2 \frac{\pi}{2M}} + \frac{1}{\sin^2 \frac{3\pi}{2M}} + \frac{1}{\sin^2 \frac{5\pi}{2M}} + \cdots + \frac{1}{\sin^2 \frac{(M-1)\pi}{2M}}\right) = 1$$

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- Apply inequality $\cot^2 \alpha < \dfrac{1}{\alpha^2} < \csc^2 \alpha$, and then let $M \to \infty$:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots = \frac{\pi^2}{8}.$$