Recent advances in first-passage percolation

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Based on joint works with

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The model

FPP is a model for a random metric on a graph.

Take the cubic lattice $(\mathbb{Z}^d, \mathcal{E}^d)$, $d \geq 2$ and assign nonnegative weights

$$(t_e)_{e \in \mathcal{E}^d} \text{ i.i.d.}$$

A lattice path γ has

$$T(\gamma) = \sum_{e \in \gamma} t_e$$

and the induced metric is

$$T(x, y) = \inf_{\gamma : x \to y} T(\gamma) \ .$$

T is a pseudometric on $\mathbb{Z}^d$:

$$T(x,y) \geq 0$$

$$T(x,y) \leq T(x,z) + T(z,y)$$

If $\mathbb{P}(t_e = 0) = 0$ then

$$T(x,y) = 0 \Rightarrow x = y.$$

The ball at the origin in this metric space is

$$B(t) = \{x \in \mathbb{Z}^d : T(0,x) \leq t\} \ .$$

Asymptotics of T : shape theorem

Extend to $T(x,y)$ for $x,y \in \mathbb{R}^d$ by choosing nearest lattice points.

Theorem (Richardson, Cox-Durrett, Kesten)

There is a non-random *limit shape* $\mathcal{B}$ such that

$$B(t)/t \to \mathcal{B} \text{ as } t \to \infty \ .$$

That is, for each $\epsilon > 0$

$$\mathbb{P}\Big((1-\epsilon)\mathcal{B} \subset B(t)/t \subset (1+\epsilon)\mathcal{B} \text{ for all large } t\Big) = 1 \ .$$

Properties of $\mathcal{B}$

It is convex, symmetric about the axes with nonempty interior.

If $\mathbb{P}(t_e = 0) < p_c(d)$ then

$\mathcal{B}$ is compact and

$\mathcal{B}$ is the unit ball of a norm g and

$$T(0,x) = g(x) + o(\|x\|) \ .$$

$\mathcal{B}$ depends on the distribution of (t_e).

Main lines of research

1. What is the set of all limit shapes?
2. What is the rate of convergence / fluctuations?
3. What is the geometry of time-minimizing paths (geodesics)?

Also max-flow min-cut problems (not my area, but see Kesten, Cerf, Théret, Rossignol, Zhang).

Question 1: Set of limit shapes

If t_e has a continuous distribution one expects:

$\mathcal{B}$ is not a Euclidean ball, but $\partial\mathcal{B}$ is positively curved.

$\mathcal{B}$ has no “flat edges” or “corners.”

$\mathcal{B}$ is not a polygon (has infinitely many extreme points).

What is known: (continuous distribution)

For d large and a large class of passage times, Kesten (1986) showed $\mathcal{B}$ is not a Euclidean ball.

Open problem: show $\mathcal{B}$ cannot be a diamond (ℓ^1 ball) for any d.

What is known: (continuous distribution)

Damron-Hochman (2010) gave examples with arbitrarily many extreme points ($d = 2$).

Open problem: give examples with infinitely many extreme points.

Nothing is known about curvature.

What is known: (non-continuous distribution)

For special distributions a bit more is known. Suppose $d = 2$ and

$$\mathbb{P}(t_e = 1) \geq \vec{p}_c, \ \mathbb{P}(t_e < 1) = 0 \ .$$

Durrett-Liggett (1981) showed that $\mathcal{B}$ has “flat edges” related to oriented percolation.

Marchand (2002) showed $\mathcal{B}$ has at least 8 extreme points.

What is known: (non-continuous distribution)

(Auffinger-Damron, 2011) $\mathcal{B}$ has infinitely many extreme points: non-polygonal.

Proof follows from differentiability of $\partial\mathcal{B}$ at the ends of the flat edges.

Open problem: show that outside the Durrett-Liggett flat edges, $\mathcal{B}$ has no corners or flat edges.

The translation-ergodic case is completely solved.

Theorem (Häggstrom-Meester, 1995)
Given a compact convex subset S of $\mathbb{Z}^d$ that is symmetric about the axes, there is an edge-distribution for (t_e) such that $\mathcal{B} = S$.

These distributions have long-range correlations and are unlikely to be adapted to the i.i.d. case.

A. Krishnan (next talk) will discuss a new approach to the limit shape, through Hamilton-Jacobi PDE's.

Question 2: Convergence rate of $B(t)/t$ to $\mathcal{B}$

Equivalent question: how large is the second term in

$$T(0,x) = g(x) + o(\|x\|) \ ?$$

Split into

$$T(0,x) - \mathbb{E}T(0,x) \text{ and } \mathbb{E}T(0,x) - g(x) \ ,$$

the *random fluctuations* and *non-random fluctuations*.

Random fluctuations: what is expected

Physicists predict that

$$\text{Var } T(0,x) \sim \|x\|^{2\chi} \text{ as } x \to \infty$$

for some d-dependent *fluctuation exponent* χ.

What does $\sim$ mean? Unclear!

Random fluctuations: what is expected

d	χ
1	$1/2$
2	$1/3$
3	?
$\cdot$	$\cdot$
d_c ?	0

χ should decrease in dimension, but it is not clear if $\chi > 0$ for all d.

Random fluctuations: what is known:

$d = 1$: $\chi = 1/2$ (for any reasonable definition).

$d \geq 2$: (Kesten, 1993)

$$0 \leq \chi \leq 1/2 \ .$$

This is the best currently known! He showed

$$C_1 \leq \mathsf{Var}\ T(0, x) \leq C_2 \|x\| \ .$$

Refinements of χ bounds:

For $d = 2$:

$$C \log \|x\| \leq \text{Var } T(0, x) \ .$$

(Newman-Piza, 1995) continuous distribution, some non-continuous.

(Auffinger-Damron, 2011) for Durrett-Liggett distributions, outside the flat edge.

Open problem: show divergence of Var $T(0, x)$ for some $d \geq 3$.

For $d \geq 2$:

$$\text{Var } T(0,x) \leq C \frac{\|x\|}{\log \|x\|} \ .$$

(Benjamini-Kalai-Schramm, 2003) for $t_e \in \{a, b\}$, equal probability.

(Benaïm-Rossignol, 2008) for t_e with continuous density on an interval that is "nearly Gamma."

(Damron-Hanson-Sosoe, 2013) all (t_e) with $\mathbb{E} t_e^2 (\log t_e)_+ < \infty$.

Newest result on exponential concentration, $d \geq 2$:

Theorem (Damron-Hanson-Sosoe, 2014)
Assume $\mathbb{E}e^{\alpha t_e} < \infty$ for some $\alpha > 0$ and $\mathbb{P}(t_e = 0) < p_c$. Then

$$\mathbb{P}\left(T(0,x) - \mathbb{E}T(0,x) \geq \lambda\sqrt{\frac{\|x\|}{\log\|x\|}}\right) \leq e^{-c\lambda}\ .$$

If only $\mathbb{E}t_e^2(\log t_e)_+ < \infty$ then

$$\mathbb{P}\left(T(0,x) - \mathbb{E}T(0,x) \leq -\lambda\sqrt{\frac{\|x\|}{\log\|x\|}}\right) \leq e^{-c\lambda}\ .$$

Extends Benaïm-Rossignol, who proved it for Nearly Gamma.

Non-random fluctuations: what is known

Suppose for some exponent γ,

$$\mathbb{E}T(0,x) - g(x) \sim \|x\|^{\gamma} \ .$$

We could precisely define upper and lower exponents

$$\overline{\gamma} = \limsup_{x \to \infty} \frac{\log(T(0,x) - \mathbb{E}T(0,x))}{\log \|x\|}$$

and $\underline{\gamma}$ similarly with $\liminf$.

Then $-1 \leq \underline{\gamma} \leq \overline{\gamma} \leq 1/2$ is the best current bound.

The history for $d \geq 2$:

(Kesten, 1993) if $\mathbb{E}e^{\alpha t_e} < \infty$ for some $\alpha > 0$,

$$-1 \leq \underline{\gamma} \leq \overline{\gamma} \leq 5/6 \ .$$

(Alexander, 1997) if $\mathbb{E}e^{\alpha t_e} < \infty$ for some $\alpha > 0$, then $\overline{\gamma} \leq 1/2$:

$$\mathbb{E}T(0,x) - g(x) \leq C(\|x\| \log^2 \|x\|)^{1/2} \ .$$

All methods require exponential bounds on random fluctuations.

Slight improvement (in preparation):

(Damron-Kubota, 2014) If $\mathbb{E}t_e^{\frac{2}{d}+\alpha} < \infty$ for some $\alpha > 0$ then

$$\mathbb{E}T(0,x) - g(x) \leq C(\|x\| \log \|x\|)^{1/2} \ .$$

A bound of this order was already found in the coordinate directions by Rhee, 1995 under $\mathbb{E}t_e^2 < \infty$.

But what is the real value of γ? $(d \geq 2)$

Assuming existence of a strongly defined exponent χ, both Alexander and Chatterjee (2012) proved

$$\overline{\gamma} \leq \chi \ .$$

(Auffinger-Damron-Hanson, 2014, in preparation) Assuming existence of a weaker exponent χ, one has a directional version of:

$$\begin{cases} \underline{\gamma} \geq \chi & \text{if } \chi < 1/2 \\ \overline{\gamma} \geq \chi & \text{if Var } T(0,x) = o\left(\frac{\|x\|}{\log \|x\|}\right) \end{cases} .$$

Weaker exponent assumption is: for some $p > 2$,

$$\underline{\chi}_2 = \overline{\chi}_p \ ,$$

where

$$\underline{\chi}_p = \liminf_n \frac{\log \|T(0, ne_1) - \mathbb{E}T(0, ne_1)\|_p}{\log n},$$

and $\overline{\chi}_p$ with $\limsup$.

(Implied by Chatterjee's exponents, which require exponential tightness.)

Question 3: Geometry of geodesics

A path $\gamma : x \to y$ such that $T(\gamma) = T(x, y)$ is a *finite geodesic*.

If $\mathbb{P}(t_e = 0) \neq p_c$ (critical probability for percolation) then

$$\mathbb{P}(\exists \text{ a geodesic from } x \text{ to } y) = 1 \text{ for all } x, y \ .$$

Open problem: is it true for $= p_c$?

If t_e has continuous distribution, there are unique geodesics.

Infinite geodesics

An infinite path each of whose finite segments are finite geodesics is an infinite geodesic.

Existence: take a sub sequential limit of geodesics $\gamma_n : 0 \to ne_1$.

Open problem: show a limit exists a.s. Known only in half-plane (Auffinger-Damron-Hanson, 2013).

Conjectures and conditional results

Under the unproven assumption of uniformly positive curvature of $\mathcal{B}$ and exponential moments, Licea-Newman, 1995 proved:

1. A.s., each infinite geodesic has a direction.
2. For any $\theta \in [0, 2\pi]$
    - there is a.s. an infinite geodesic with direction θ and
    - $(d = 2)$ if θ is not in a deterministic Lebesgue null set D, a.s., any two infinite geodesics with direction θ coalesce.

In particular (conjecturally),

a.s. there are infinitely many disjoint infinite geodesics and

each has an asymptotic direction .

Also there should be a.s. no “doubly infinite” geodesics. There are partial results by Licea-Newman under curvature assumption.

What is known: number of geodesics.

Consider the union of all finite geodesics. The number of topological ends, N, of this graph is a measure of the number of distinct infinite geodesics.

(Hoffman, 2005, Garet-Marchand, 2005) $N \geq 2$.

(Hoffman, 2008) $N \geq 4$ for $d = 2$..

(Damron-Hochman, 2010, Auffinger-Damron, 2011) $d = 2$, for some (t_e) distributions, $N = \infty$.

Open problem: show $N = \infty$ for all weight distributions.

What is known: directions (very little!)

Open problem: show that there is a.s. an infinite geodesic with an asymptotic direction.

Theorem (Damron-Hanson, 2012)

For $d = 2$, assume $\mathbb{E}t_e^{\frac{1}{d}+\alpha} < \infty$ for some $\alpha > 0$. There are infinite geodesics directed in sectors of aperture $\leq \pi/2$.

These sectors are related to the limit shape.

∂B is differentiable a.e., so let L be a tangent line. The sector of angles S of contact of L and $\mathcal{B}$ form such a sector.

So there is an infinite geodesic directed in S.

What about the other conjectures (coalescence, bigeodesics)?

Progress made in Damron-Hanson, 2012, on subsequential limits. Let L be a non-horizontal line in $\mathbb{R}^2$ and for $r > 0$ set

$$L_r = L + re_1 \ .$$

For each r there is a "geodesic graph" G_r with vertex set $\mathbb{Z}^2$ and (directed) edge set

$$E_r = \{e : e \text{ is in a geodesic from some } x \to L_r\} \ .$$

Theorem (Damron-Hanson, 2012)

Assume $\mathbb{E}t_e^{\frac{1}{d}+\alpha} < \infty$ for some $\alpha > 0$ and t_e has continuous distribution. A.s., there is a random sequence $r_n \to \infty$ such that the graphs G_{r_n} converge to some G with the following properties.

1. From each x emanates exactly one infinite directed path Γ_x.
2. For x, y, Γ_x and Γ_y coalesce.
3. For each x, the backward cluster $C_x = \{y : x \in \Gamma_y\}$ is finite.

All Γ_x's are infinite geodesics directed in a sector.

Idea of proofs is to focus on Busemann functions:

$$B_r(x,y) = T(x, L_r) - T(y, L_r)$$

and define a “gradient field”

$$\{B_r(x,y) : x, y \text{ neighbors}\} .$$

$\mathbb{P}$ induces a measure on gradient fields and we extract a translation-invariant sub sequential limit in distribution as $r \to \infty$. (Such fields are examples of “stationary cocyles” from the recent work of Georgiou-Rassoul-Agha-Seppälainen on LPP.)

Auffinger, Damron. (2011, PTRF). Differentiability at the edge of the percolation cone and related results in FPP.

Auffinger, Damron, Hanson. (2013, AAP). Limiting geodesics for FPP on subsets of $\mathbb{Z}^2$.

Damron, Hanson. (2012, CMP). Busemann functions and infinite geodesics in $2d$ FPP.

Damron, Hanson, Sosoe. (2013, preprint). Sublinear variance in FPP for general distributions.

Damron, Hanson, Sosoe. (2014, preprint). Subdiffusive concentration in FPP.

Damron, Hochman. (2010, AAP). Examples of non-polygonal limit shapes in i.i.d. FPP and infinite coexistence in spatial growth models.

Works in progress with Auffinger, Hanson, Kubota.