Hölder continuity for the nonlinear stochastic heat equation with rough initial conditions

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Stochastic Heat Equation (SHE)

$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{\nu}{2}\frac{\partial^2}{\partial x^2}\right) u(t,x) = \rho(u(t,x))\,\dot{W}(t,x), \quad x \in \mathbb{R},\ t \in \mathbb{R}_+^*, \\ u(0,\cdot) = \mu(\cdot)\ , \end{cases} \tag{SHE}$$

- $\dot{W}$ is the space-time white noise;
- ρ is Lipschitz continuous;
- μ is the initial measure (to be specified).

$$u(t,x) = J_0(t,x) + \iint_{[0,t]\times\mathbb{R}} \rho(u(s,y)) G_\nu(t-s, x-y) W(\mathrm{d}s, \mathrm{d}y).$$

$$G_\nu(t,x) = \frac{1}{\sqrt{2\pi\nu t}} \exp\left(-\frac{x^2}{2t}\right)$$

$$J_0(t,x) := (\mu * G_\nu(t,\cdot))(x)$$

Definition of random field solution

$$u(t,x) = J_0(t,x) + \underbrace{\iint_{[0,t]\times\mathbb{R}} \rho\left(u(s,y)\right) G_\nu(t-s,x-y) W(\mathrm{d}s,\mathrm{d}y)}_{:=I(t,x)} . \quad \text{(SHE)}$$

Definition (Random field solution)

$u = \left(u(t,x) : (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\right)$ is called a *random field solution* to (SHE) if

(1) u is adapted, i.e., for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $u(t,x)$ is $\mathcal{F}_t$-measurable;

(2) u is jointly measurable with respect to $\mathcal{B}\left(\mathbb{R}_+^* \times \mathbb{R}\right) \times \mathcal{F}$;

(3) $\left(G_\nu^2 \star ||\rho(u)||_2^2\right)(t,x) < +\infty$ for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, and $(t,x) \mapsto I(t,x) : \mathbb{R}_+^* \times \mathbb{R} \mapsto L^2(\Omega)$ is continuous;

(4) u satisfies (SHE) almost surely, for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$.

$$\left(G_\nu^2 \star ||\rho(u)||_2^2\right)(t,x) := \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \, G_\nu^2(t-s,x-y) \, ||\rho(u(s,y))||_2^2 .$$

Rough initial data

- Initial data has a bounded density function (Walsh theory [2]), (Bounded initial data) i.e., $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$ with $f \in L^\infty(\mathbb{R})$.
- Measure-valued initial data (Ch. & Dalang [1]).

$$\mathcal{M}_H(\mathbb{R}) := \left\{\text{signed Borel meas. } \mu, \text{ s.t. } \int_{\mathbb{R}} e^{-ax^2}|\mu|(\mathrm{d}x) < +\infty,\ \forall a > 0\right\}$$

$$\left(|\mu| * G_\nu(t,\cdot)\right)(x) := \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\nu t}} e^{-\frac{(x-y)^2}{2\nu t}} |\mu|(\mathrm{d}y) < +\infty \quad \forall t > 0, \forall x \in \mathbb{R}.$$

Initial data cannot go beyond measures. No random field solution for δ_0'.

[1] L. Chen, R. Dalang, Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions, *Ann. Probab.*, (accepted, pending revision), 2014.

[2] J. B. Walsh. An introduction to stochastic partial differential equations. In: *École d'été de probabilités de Saint-Flour, XIV—1984*, pp. 265–439. Springer, Berlin, 1986.

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$$J_0(t,x) \in C^\infty(\mathbb{R}_+^* \times \mathbb{R})$$

$$I(t,x) \in C_{?,?}(\mathbb{R}_+^* \times \mathbb{R})$$

Some notation for locally Hölder continuous functions

Given a subset $D \subseteq \mathbb{R}_+ \times \mathbb{R}$ and positive constants β_1, β_2, denote by $C_{\beta_1,\beta_2}(D)$ the set of functions $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ with the following property:

For each compact subset $\tilde{D} \subset D$, $\exists C$ s.t. for all (t, x) and $(s, y) \in \tilde{D}$,

$$|v(t,x) - v(s,y)| \leq C\left(|t-s|^{\beta_1} + |x-y|^{\beta_2}\right).$$

$$C_{\beta_1-,\beta_2-}(D) := \bigcap_{0<\alpha_1<\beta_1} \bigcap_{0<\alpha_2<\beta_2} C_{\alpha_1,\alpha_2}(D)\,.$$

$$u(t,x) = J_0(t,x) + I(t,x)$$

$$\mathcal{M}_H(\mathbb{R}) := \left\{ \text{signed Borel meas. } \mu, \text{ s.t. } \int_{\mathbb{R}} e^{-ax^2} |\mu|(\mathrm{d}x) < +\infty, \ \forall a > 0 \right\}$$

Theorem

(1) *If $\mu \in \mathcal{M}_H(\mathbb{R})$, then $I \in C_{\frac{1}{4}-,\frac{1}{2}-}(\mathbb{R}_+^* \times \mathbb{R})$ a.s. Therefore,*

$$u \in C_{\frac{1}{4}-,\frac{1}{2}-}(\mathbb{R}_+^* \times \mathbb{R}), \quad a.s.$$

$$u(t,x) = J_0(t,x) + I(t,x)$$

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$$\mathcal{M}_H^*(\mathbb{R}) := \left\{ \mu(\mathrm{d}x) = f(x)\mathrm{d}x, \text{ s.t. } \exists a \in \left]1,2\right[\,, \sup_{x \in \mathbb{R}} |f(x)| e^{-|x|^a} < +\infty \right\}.$$

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$$u \in C_{\frac{1}{4}-,\frac{1}{2}-}(\mathbb{R}_+^* \times \mathbb{R}), \quad a.s.$$

(2) *If $\mu \in \mathcal{M}_H^*(\mathbb{R})$ with $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$, then $I \in C_{\frac{1}{4}-,\frac{1}{2}-}(\mathbb{R}_+ \times \mathbb{R})$, a.s. Moreover,*
(i) If f is continuous, then

$$u \in C(\mathbb{R}_+ \times \mathbb{R}) \cap C_{\frac{1}{4}-,\frac{1}{2}-}(\mathbb{R}_+^* \times \mathbb{R}), \quad a.s.$$

(ii) If f is α-Hölder continuous, then

$$u \in C_{(\frac{\alpha}{2} \wedge \frac{1}{4})-,(\alpha \wedge \frac{1}{2})-}(\mathbb{R}_+ \times \mathbb{R}) \cap C_{\frac{1}{4}-,\frac{1}{2}-}(\mathbb{R}_+^* \times \mathbb{R}), \quad a.s.$$

Difficulties with rough initial data

Conventional method: For $p > 1$ and $q = p/(p-1)$, $t < t'$ ($\rho(u) = u$), Set

$$\mathcal{G}_\nu(t-s, x-y; t'-s, x'-y) = G_\nu(t-s, x-y) - G_\nu(t'-s, x'-y).$$

$$\begin{aligned}
||I(t,x) - I(t',x')||_{2p}^{2p} &= \left|\left|\iint_{[0,t']\times\mathbb{R}} \mathcal{G}_\nu(t-s, x-y; t'-s, x'-y) u(s,y) W(\mathrm{d}s\mathrm{d}y)\right|\right|_{2p}^{2p} \\
&\le C \left[\int_0^{t'}\int_{\mathbb{R}} \mathcal{G}_\nu(\cdots)^2 \mathrm{d}s\mathrm{d}y\right]^{p/q} \int_0^{t'}\int_{\mathbb{R}} \mathcal{G}_\nu^2 \cdot \left(1 + ||u(s,y)||_{2p}^{2p}\right) \mathrm{d}s\mathrm{d}y \\
&\le C \sup_{s\in[0,t']} \sup_{y\in\mathbb{R}} \left(1 + ||u(s,y)||_{2p}^{2p}\right) \left[\int_0^{t'} \int_{\mathbb{R}} \mathcal{G}_\nu(\cdots)^2 \mathrm{d}s\mathrm{d}y\right]^p \\
&\le C \sup_{s\in[0,t']} \sup_{y\in\mathbb{R}} \left(1 + ||u(s,y)||_{2p}^{2p}\right) \left[|t'-t|^{p/2} + |x'-x|^p\right]
\end{aligned}$$

[1] Robert C. Dalang. *The stochastic wave equation.* In *A minicourse on stochastic partial differential equations*, volume 1962 of Lecture Notes in Math. Springer, Berlin, 2009.
[2] Marta Sanz-Solé and Mònica Sarrà. *Hölder continuity for the stochastic heat equation with spatially correlated noise.* In *Seminar on Stochastic Analysis, Random Fields and Applications, III*, volume 52 of *Progr. Probab.*. Birkhäuser, Basel, 2002.
[3] Tokuzo Shiga. *Two contrasting properties of solutions for one-dimensional stochastic partial differential equations.* Canad. J. Math., 46(2):415–437, 1994.

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&\le C \sup_{s\in[0,t']}\sup_{y\in\mathbb{R}} \left(1 + ||u(s,y)||_{2p}^{2p}\right)\left[\int_0^{t'}\int_{\mathbb{R}} \mathcal{G}_\nu(\cdots)^2 \mathrm{d}s\mathrm{d}y\right]^p \\
&\le C \sup_{s\in[0,t']}\sup_{y\in\mathbb{R}} \left(1 + ||u(s,y)||_{2p}^{2p}\right)\left[|t'-t|^{p/2} + |x'-x|^p\right]
\end{aligned}$$

Tails $\Rightarrow$ integrability of x at $\pm\infty$.
Measure $\Rightarrow$ integrability of t at 0: e.g., $\mu = \delta_0$,

$$||u(s,y)||_{2p}^{2} \ge ||u(s,y)||_2^2 \ge G_{\frac{\nu}{2}}(s,y)\frac{1}{\sqrt{4\pi\nu s}} = \frac{C}{s} e^{-\frac{y^2}{\nu s}} \Rightarrow p < 3/2.$$

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\end{aligned}$$

Lemma. For each $K_n := [1/n, n] \times [n, n]$ and $p \ge 2$, find $C_{n,p}$ such that

$$\left|\left|I(t,x) - I(t',x')\right|\right|_p \le C_{n,p} \left(|t-t'|^{1/4} + |x-x'|^{1/2}\right), \quad \forall (t,x), (t',x') \in K_n.$$

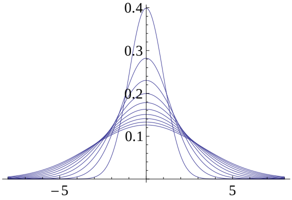
Instead of

$$\iint_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}s\mathrm{d}y \left(G_\nu(t-s,x-y) - G_\nu(t'-s,x'-y)\right)^2 \leq C\left(|x-x'| + \sqrt{|t-t'|}\right).$$

For all (t,x) and $(t',x') \in [1/n, n] \times [-n, n]$, find $C_n > 0$ s.t.,

$$\iint_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}s\mathrm{d}y\, J_0(s,y)^2 \left(G_\nu\left(t-s,x-y\right) - G_\nu(t'-s,x'-y)\right)^2 \leq C_n\left(|x-x'| + \sqrt{|t-t'|}\right).$$

Two key estimates on heat kernel



$$G_\nu(t,x) = \frac{1}{\sqrt{2\pi\nu t}} \exp\left(-\frac{x^2}{2t}\right)$$

Lemma 1. For all $L > 0$, $0 < \beta < 1$, $t > 0$, $x \in \mathbb{R}$, and $|h| \le \beta L$, $\exists C \approx 0.45$,

$$|G_\nu(t,x+h) + G_\nu(t,x-h) - 2G_\nu(t,x)|$$
$$\le 2|h| \left(\frac{C}{\sqrt{2\nu t}} + \frac{1}{(1-\beta)L}\right) \left[G_\nu(t,x) + e^{\frac{3L^2}{2\nu t}} \left\{G_\nu(t,x-2L) + G_\nu(t,x+2L)\right\}\right].$$

Lemma 2. For all $t > 0$, $n > 1$, $x \in \mathbb{R}$ and $0 < r < n^2 t$,

$$\left|G_{\frac{\nu}{2}}(t+r,x) - G_{\frac{\nu}{2}}(t,x)\right| \le \frac{3}{2}\frac{\sqrt{1+n^2}}{\sqrt{t}}\, G_{\frac{\nu(1+n^2)}{2}}(t,x)\sqrt{r}.$$

Moment formula

$$||u(t,x)||_p^2 \leq J_0^2(t,x) + \left(J_0^2 \star \mathcal{K}_p(t,x)\right)(t,x)$$

$$\mathcal{K}(t,x;\lambda) := G_{\frac{\nu}{2}}(t,x)\left(\frac{\lambda^2}{\sqrt{4\pi\nu t}} + \frac{\lambda^4}{2\nu} e^{\frac{\lambda^4}{4\nu}} \Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right)\right)$$

$$\mathcal{K}_p(t,x) := \mathcal{K}(t,x;4\sqrt{p}\,L_\rho)$$

[1] L. Chen, R. Dalang, Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions, *Ann. Probab.*, (accepted, pending revision), 2014.

Related work

$$u \in C_{\frac{1}{4}-,\frac{1}{2}-}(\mathbb{R}_+^* \times \mathbb{R})$$

- Bounded initial data (Walsh theory).
- Shiga's work $f \in C_{tem}(\mathbb{R})$: continuous function with tails tails grow at most exponentially at $\pm\infty$.
- Sanz-Solé & Sarrà's work on SHE over $\mathbb{R}^d$ with spatially homogeneous colored noise which is white in time: bounded continuous function.
- Work by Conus *et al*: finite measure, $1/2-$ in space.
- Work by Dalang, Khoshnevisan & Nualart: a system of SHE's with vanishing initial data.

- SHE on bounded domains rather than $\mathbb{R}$: Maximal inequality and stochastic convolutions. (initial data in some Banach space)

J. B. Walsh. An introduction to stochastic partial differential equations. In: *École d'été de probabilités de Saint-Flour, XIV—1984* , pp. 265–439. Springer, Berlin, 1986.

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M. Sanz-Solé and M. Sarrà. Hölder continuity for the stochastic heat equation with spatially correlated noise. In: *Seminar on Stochastic Analysis, Random Fields and Applications, III* , pp. 259–268. Birkhäuser, Basel, 2002. (R. C. Dalang, M. Dozzi and F. Russo, eds).

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D. Conus, M. Joseph, D. Khoshnevisan, and S.-Y. Shiu. Initial measures for the stochastic heat equation. *Ann. Inst. Henri Poincaré Probab. Stat.*, 2014.

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R. C. Dalang, D. Khoshnevisan, and E. Nualart. Hitting probabilities for systems for non-linear stochastic heat equations with multiplicative noise. *Probab. Theory Related Fields*, 2009.

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Z. Brzeźniak. On stochastic convolution in Banach spaces and applications. *Stochastics Stochastic Rep.* 61(3-4):245–295, 1997.
S. Peszat and J. Seidler. Maximal inequalities and space-time regularity of stochastic convolutions. *Mathematica Bohemica* 123(1): 7-32, 1998.

Thank you!

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