

Extrema of discrete 2D Gaussian Free Field and Liouville quantum gravity

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Discrete Gaussian Free Field (DGFF)

$D \subset \mathbb{R}^d$ (or \mathbb{C} in $d = 2$) bounded, open, “nice” boundary

$D_N := \{x \in \mathbb{Z}^d : x/N \in D\}$

$G_N^D :=$ Green function of SRW on D_N killed upon exit

$$G_N^D(x, y) = E^x \left(\sum_{k=0}^{\tau-1} 1_{\{X_k=y\}} \right)$$

where $\tau :=$ first exit time from D_N

Definition

DGFF := Gaussian process $\{h_x : x \in \mathbb{Z}^d\}$ with

$$E(h_x) = 0 \quad \text{and} \quad E(h_x h_y) = G_N^D(x, y)$$

Other ways to define same object

Hilbert space valued Gaussian:

$\mathcal{H} := \{f \in \ell^2(\mathbb{Z}^d) : f = 0 \text{ on } D_N^c\}$ with $\langle f, g \rangle := \sum_x \nabla f(x) \cdot \nabla g(x)$
 $\{\varphi_n\}$ ONB in \mathcal{H} , $\{Z_n\}$ i.i.d. $\mathcal{N}(0,1)$

$$h_x := \sum_{n=1}^{|D_N|} Z_n \varphi_n(x)$$

Dynamical equilibrium:

DGFF = stationary state for Glauber dynamics with transition rule

$$h_x \rightarrow Z + \frac{1}{2d} \sum_{y: |y-x|=1} h_y$$

where $Z =$ independent $\mathcal{N}(0,1)$

Other ways to define same object

Gibbs-Markov property:

$\tilde{D} \subset D$. Then

$$h^D \stackrel{\text{law}}{=} h^{\tilde{D}} + \varphi^{D, \tilde{D}}$$

where

- (1) $h^{\tilde{D}}$ and $\varphi^{D, \tilde{D}}$ independent Gaussian fields
- (2) $x \mapsto \varphi^{D, \tilde{D}}(x)$ discrete harmonic on \tilde{D}_N

Why 2D?

For x with $\text{dist}(x, D_N^c) > \delta N$,

$$\text{Var}(h_x) = G_N(x, x) \asymp \begin{cases} N, & \text{if } d = 1, \\ \log N, & \text{if } d = 2, \\ 1, & \text{if } d \geq 3. \end{cases}$$

In $d = 2$:

$$\begin{aligned} G_N(x, y) &= g \log N - a(x, y) + o(1), & N \gg 1 \\ a(x, y) &= g \log |x - y| + O(1), & |x - y| \gg 1 \end{aligned}$$

where

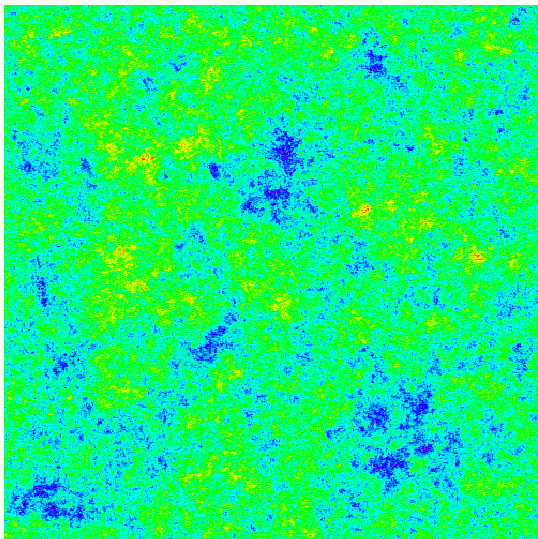
$$g := \frac{2}{\pi}$$

(In physics, $g = \frac{1}{2\pi}$) The model is **nearly scale invariant**:

$$G_{2N}(2x, 2y) = G_N(x, y) + o(1)$$

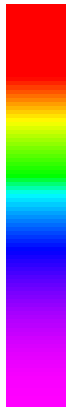
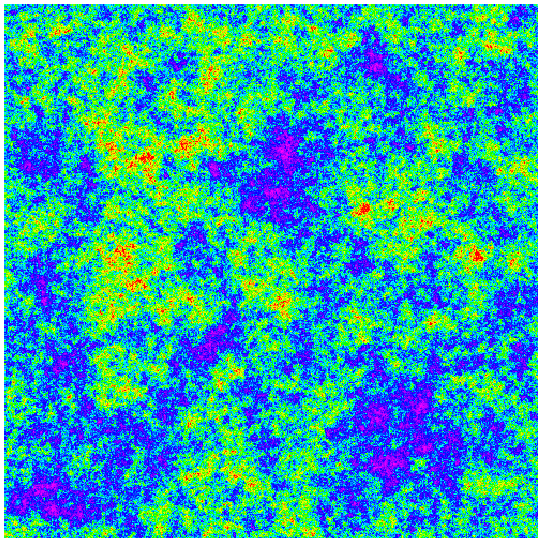
DGFF on 500×500 square

Uniform color system



DGFF on 500×500 square

Skewed color system



Some known facts

Absolute maximum

Setting and notation: $D_N := (0, N)^2 \cap \mathbb{Z}^2$

$$M_N := \max_{x \in D_N} h_x \quad \text{and} \quad m_N := EM_N$$

Leading scale (Bolthausen, Deuschel & Giacomin):

$$m_N \sim 2\sqrt{g} \log N$$

Tightness for a subsequence (Bolthausen, Deuschel & Zeitouni):

$$2E|M_N - m_N| \leq m_{2N} - m_N$$

Full tightness (Bramson & Zeitouni):

$$EM_N = 2\sqrt{g} \log N - \frac{3}{4}\sqrt{g} \log \log N + O(1)$$

Convergence in law (Bramson, Ding & Zeitouni)

Some known facts

Extreme points

Extreme level set:

$$\Gamma_N(t) := \{x \in D_N : h_x \geq m_N - t\}$$

Extreme point tightness (Ding & Zeitouni):

$$\exists c, C \in (0, \infty) : \lim_{\lambda \rightarrow \infty} \liminf_{N \rightarrow \infty} P(e^{c\lambda} \leq |\Gamma_N(\lambda)| \leq e^{C\lambda}) = 1$$

and $\exists c > 0$ s.t.

$$\lim_{r \rightarrow \infty} \limsup_{N \rightarrow \infty} P(\exists x, y \in \Gamma_N(c \log \log r) : r \leq |x - y| \leq N/r) = 0$$

Extreme point process

Full process: Measure η_N on $\overline{D} \times \mathbb{R}$

$$\eta_N := \sum_{x \in D_N} \delta_{x/N} \otimes \delta_{h_x - m_N}$$

Problem: Values in one peak strongly correlated

Local maxima only: $\Lambda_r(x) := \{z \in \mathbb{Z}^2 : |z - x| \leq r\}$

$$\eta_{N,r} := \sum_{x \in D_N} \mathbf{1}_{\{h_x = \max_{z \in \Lambda_r(x)} h_z\}} \delta_{x/N} \otimes \delta_{h_x - m_N}$$

Main results

Extreme process convergence

Theorem (Convergence to Cox process)

There is a random Borel measure Z^D on \bar{D} with $0 < Z^D(\bar{D}) < \infty$ a.s. such that for any $r_N \rightarrow \infty$ and $N/r_N \rightarrow \infty$,

$$\eta_{N,r_N} \xrightarrow[N \rightarrow \infty]{\text{law}} \text{PPP}\left(Z^D(dx) \otimes e^{-\alpha h} dh\right)$$

where $\alpha := 2/\sqrt{g} = \sqrt{2\pi}$.

Asymptotic law of maximum: Setting $Z := Z^D(\bar{D})$,

$$P(M_N \leq m_N + t) \xrightarrow{N \rightarrow \infty} E(e^{-\alpha^{-1} Z e^{-\alpha t}})$$

Joint law position/value: $A \subset D$ open, $\hat{Z}(A) = Z^D(A) / Z^D(\bar{D})$

$$P(M_N \leq m_N + t, N^{-1} \operatorname{argmax}(h) \in A) \xrightarrow{N \rightarrow \infty} E(\hat{Z}(A) e^{-\alpha^{-1} Z e^{-\alpha t}})$$

In fact: Key steps of the proof

Proof of Theorem

Distributional invariance

Note: $\{\eta_{N,r_N} : N \geq 1\}$ tight, can extract converging subsequences
Denote $\langle \eta, f \rangle := \int \eta(dx, dh) f(x, h)$

Proposition (Distributional invariance)

Suppose η = a weak-limit point of some $\{\eta_{N_k, r_{N_k}}\}$. Then for any $f : D \times \mathbb{R} \rightarrow [0, \infty)$ continuous, compact support,

$$E(e^{-\langle \eta, f \rangle}) = E(e^{-\langle \eta, f_t \rangle}), \quad t > 0,$$

where

$$f_t(x, h) := -\log E e^{-f(x, h + B_t - \frac{\alpha}{2} t)}$$

with $B_t :=$ standard Brownian motion.

Proposition explained

We may write

$$\eta = \sum_{i \geq 1} \delta_{x_i, h_i}$$

Letting $\{B_t^{(i)}\}$ be independent standard Brownian motions, set

$$\eta_t := \sum_{i \geq 1} \delta_{x_i, h_i + B_t^{(i)} - \frac{\alpha}{2} t}$$

Well defined as $t \mapsto |\Gamma_N(t)|$ grows only exponentially. Then

$$E(e^{-\langle \eta_t, f \rangle}) = E(e^{-\langle \eta, f_t \rangle})$$

and so Proposition says

$$\eta_t \stackrel{\text{law}}{=} \eta, \quad t > 0$$

Proof of Proposition 1

Gaussian interpolation: $h', h'' \stackrel{\text{law}}{=} h$, independent

$$h \stackrel{\text{law}}{=} \left(1 - \frac{t}{g \log N}\right)^{1/2} h' + \left(\frac{t}{g \log N}\right)^{1/2} h''$$

Now let x be such that $h'_x \geq m_N - \lambda$. Then

$$\begin{aligned} \left(1 - \frac{t}{g \log N}\right)^{1/2} h'_x &= h'_x - \frac{1}{2} \frac{t}{g \log N} h'_x + o(1) \\ &= h'_x - \frac{t}{2} \frac{m_N}{g \log N} + o(1) \\ &= h'_x - \frac{\alpha}{2} t + o(1) \end{aligned}$$

Proof of Proposition II

Concerning h'' , denote

$$\tilde{h}_x'' := \left(\frac{t}{g \log N} \right)^{1/2} h_x''$$

By properties of G_N^D we have

$$\text{Cov}(\tilde{h}_x'', \tilde{h}_y'') = \begin{cases} t + o(1), & \text{if } |x - y| \leq r \\ o(1), & \text{if } |x - y| \geq N/r \end{cases}$$

So we conclude:

$$\left\{ \tilde{h}_x'' : x \in D_L, h'_x \geq m_N - \lambda, h'_x = \max_{z \in \Lambda_r(x)} h'_z \right\}$$

has asymptotically the law of $\{B_t^{(i)}\}$. □

Proof of Theorem

Question before the house

Question: Characterize the point processes that are invariant under independent Dysonization

$$x \mapsto x + B_t - \frac{\alpha}{2}t$$

of the points?

Easy to check: $\text{PPP}(v(dx) \otimes e^{-\alpha h} dh)$ okay for any v (even random)

Any other solutions?

Liggett's 1977 derivation

For $t > 0$ define Markov kernel $(Pg)(x, h) := E^0 g(x, h + B_t - \frac{\alpha}{2}t)$
Set $g(x, h) := e^{-f(x, h)}$ for $f \geq 0$ continuous with compact support.
Proposition implies

$$E(e^{-\langle \eta, f \rangle}) = E(e^{-\langle \eta, f^{(n)} \rangle})$$

where

$$f^{(n)}(x, h) = -\log(P^n e^{-f})(x, h)$$

P has **uniform dispersivity property**: For $C \subset \overline{D} \times \mathbb{R}$ compact

$$\sup_{x, h} P^n((x, h), C) \xrightarrow{n \rightarrow \infty} 0$$

and thus $P^n e^{-f} \rightarrow 1$ uniformly. Expanding the log,

$$f^{(n)} \sim 1 - P^n e^{-f} \quad \text{as } n \rightarrow \infty$$

Liggett's 1977 derivation (continued)

Hence

$$E(e^{-\langle \eta, f \rangle}) = \lim_{n \rightarrow \infty} E(e^{-\langle \eta, 1 - P^n e^{-f} \rangle}) \quad (*)$$

But, as P is Markov,

$$\langle \eta, 1 - P^n e^{-f} \rangle = \langle \eta P^n, 1 - e^{-f} \rangle$$

(*) shows that $\{\eta P^n : n \geq 1\}$ is tight. Along a subsequence

$$\eta P^{n_k}(dx, dh) \xrightarrow[k \rightarrow \infty]{\text{law}} M(dx, dh)$$

and so

$$E(e^{-\langle \eta, f \rangle}) = E(e^{-\langle M, 1 - e^{-f} \rangle})$$

i.e., $\eta = \text{PPP}(M(dx, dh))$. Clearly,

$$MP \stackrel{\text{law}}{=} M$$

Proof of Theorem

Question before the house II

Question: What M can we get?

Theorem (Liggett 1977)

$MP \stackrel{\text{law}}{=} M$ implies $MP = M$ a.s. when P is a kernel of

- (1) an irreducible, recurrent Markov chain
- (2) a random walk on a closed abelian group w/o proper closed invariant subset

(2) applies in our case.

Note: $MP = M$ means M invariant for the chain. Choquet-Deny (or $t \downarrow 0$) show

$$M(dx, dh) = Z^D(dx) \otimes e^{-\alpha h} dh + \tilde{Z}^D(dx) \otimes dh$$

Tightness of maximum implies $\tilde{Z}^D = 0$ a.s. □

Proof of Theorem

Finishing touches

We thus know $\eta_{N_k, r_{N_k}} \xrightarrow{\text{law}} \eta$ implies

$$\eta = \text{PPP}(Z^D(dx) \otimes e^{-\alpha h} dh)$$

for some random Z^D . But for $Z := Z^D(\bar{D})$, this reads

$$P(M_{N_k} \leq m_{N_k} + t) \xrightarrow[k \rightarrow \infty]{} E(e^{-\alpha^{-1} Z e^{-\alpha t}})$$

Hence the law of $Z^D(\bar{D})$ unique if limit law of maximum unique.

An enhanced version implies uniqueness of law of $Z^D(dx)$. \square

Properties of Z -measure

Theorem

The measure Z^D satisfies:

- (1)** $Z^D(A) = 0$ a.s. for any Borel $A \subset \bar{D}$ with $\text{Leb}(A) = 0$
- (2)** $\text{supp}(Z^D) = \bar{D}$ and $Z^D(\partial D) = 0$ a.s.
- (3)** Z^D is non-atomic a.s.

Property (3) is only barely true:

Conjecture

Z^D is supported on a set of zero Hausdorff dimension

Fancy properties

Gibbs-Markov

Recall $\tilde{D} \subset D$ yields $h^D \stackrel{\text{law}}{=} h^{\tilde{D}} + \varphi^{D, \tilde{D}}$

Fact: $\varphi^{D, \tilde{D}} \stackrel{\text{law}}{\rightarrow} \Phi^{D, \tilde{D}}$ on \tilde{D} where

(1) $\{\Phi^{D, \tilde{D}}(x) : x \in \tilde{D}\}$ Gaussian with $E\Phi^{D, \tilde{D}}(x) = 0$ and

$$\text{Cov}(\Phi^{D, \tilde{D}}(x), \Phi^{D, \tilde{D}}(y)) = G^D(x, y) - G^{\tilde{D}}(x, y)$$

(2) $x \mapsto \Phi^{D, \tilde{D}}(x)$ harmonic on \tilde{D} a.s.

Theorem (Gibbs-Markov property)

Suppose $\tilde{D} \subset D$ be such that $\text{Leb}(D \setminus \tilde{D}) = 0$. Then

$$Z^D(dx) \stackrel{\text{law}}{=} e^{\alpha \Phi^{D, \tilde{D}}(x)} Z^{\tilde{D}}(dx)$$

Fancy properties

Conformal invariance (CFI)

Theorem (Conformal invariance)

Suppose $f: D \rightarrow \tilde{D}$ analytic bijection. Then

$$Z^{\tilde{D}} \circ f(dx) \stackrel{\text{law}}{=} |f'(x)|^4 Z^D(dx)$$

In particular, for D simply connected and $\text{rad}_D(x)$ conformal radius

$$\text{rad}_D(x)^{-4} Z^D(dx)$$

is invariant under conformal maps of D .

Note:

- (1) $\text{Leb} \circ f(dx) = |f'(x)|^2 \text{Leb}(dx)$ and so $\text{rad}_D(x)^{-2} \text{Leb}(dx)$ is CFI. Consistent w/ zero-Hausdorff dimension conjecture
- (2) For D simply connected, suffices to know $Z^{\mathbb{D}}$ for $\mathbb{D} :=$ unit disc. So this is a **statement of universality**

Unifying scheme?

Continuum Gaussian Free Field

Continuum GFF := Gaussian on $H_0^1(D)$ w.r.t. norm $f \mapsto \pi \|\nabla f\|_2^2$

Formal expression: $h(x) = \sum_{n \geq 1} Z_n \phi_n(x)$

Exists only as a linear functional on $H_0^1(D)$:

$$h(f) = \sqrt{\pi} \sum_{n \geq 1} Z_n \langle \nabla f, \nabla \phi_n \rangle_{L^2(D)}$$

Derivative martingale:

$$M'(dx) = [2\text{Var}(h(x)) - h(x)] e^{2h(x) - 2\text{Var}(h(x))} dx$$

Can be defined by smooth approximations to h or expansion in ONB (Duplantier, Sheffield, Rhodes, Vargas)

KPZ relation links M' -measure of sets to Lebesgue measure

Unifying scheme?

Liouville Quantum Gravity

For D simply connected,

$$M^D(dx) := \text{rad}_D(x)^2 M'(dx)$$

This is the **Liouville Quantum Gravity** measure constructed in (Duplantier, Sheffield, Rhodes, Vargas)

Theorem (B-Louidor, in progress)

There is constant $c_\star \in (0, \infty)$ s.t. for all D

$$Z^D(dx) \stackrel{\text{law}}{=} c_\star M^D(dx)$$

Full extreme process

Recall we were interested in $\eta_N := \sum_{x \in D_N} \delta_{x/N} \otimes \delta_{h_x - m_N}$
A better representation by **cluster process** on $\bar{D} \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^2}$:

$$\hat{\eta}_{N,r} := \sum_{x \in D_N} 1_{\{h_x = \max_{z \in \Lambda_r(x)} h_z\}} \delta_{x/N} \otimes \delta_{h_x - m_N} \otimes \delta_{\{h_x - h_{x+z} : z \in \mathbb{Z}^2\}}$$

Theorem (B-Louidor, in progress)

There is a measure μ on \mathbb{Z}^2 such that (for $r_N \rightarrow \infty$, $N/r_N \rightarrow \infty$)

$$\hat{\eta}_{N,r_N} \xrightarrow[N \rightarrow \infty]{\text{law}} \text{PPP} \left(Z^D(dx) \otimes e^{-\alpha h} dh \otimes \mu(d\phi) \right)$$

where $\alpha := 2/\sqrt{g} = \sqrt{2\pi}$.

Capable of capturing **universality** w.r.t. short-range perturbations

THE END