

Global Attractors for Infinite-Dimensional Systems with Applications to the *complex Ginzburg-Landau equation*

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Abstract

Many infinite-dimensional dynamical systems are controlled by a finite number of degrees of freedom. This motivates one to study the possible existence of attractors, absorbing sets, and manifolds for the mathematical model at hand. Furthermore, for particular systems, it is of interest to explore their possible finite dimensionality. Fundamental semigroup notation will be introduced, and a detailed analysis of the theorem for the existence of a global attractor will be discussed. The *complex Ginzburg-Landau equation* will be presented as an involved example, which will be shown to possess a global attractor. Bounds for the appropriate Hausdorff dimension of a given attractor in dissipative systems will be addressed, but not derived.

1 Introduction

It has been shown that many dissipative partial differential equations (p.d.e.) or infinite-dimensional systems may be controlled by a finite number of degrees of freedom. Therefore one needs to develop the necessary tools to be able to show a particular p.d.e. may exhibit this behavior. Classical examples can be found in continuum mechanics and fluid dynamics problems where this low dimensionality has been observed theoretically, numerically, and experimentally[1]. This observation has brought a wave of exploration into finding proper estimates and bounds for the finite dimensionality.

In most cases it is not intuitively obvious that a system can behave asymptotically by a low degree of dimensionality. Therefore the ground work to explore this potential feature will be detailed, and is the primary goal of this paper. In essence, if one can show a system contains an absorbing set or, better yet, a global attractor then one can conclude the system's asymptotic state is controlled by a smaller subspace of the range, which may be finite-dimensional. Thus further analysis is needed to explore the degree of the dimension of a particular system's asymptotic state. These two issues will be detailed in **Sec. 2, 3**.

As noted before, the goal of this paper is to detail the steps to show a particular dissipative p.d.e. possesses an absorbing set and/or an attractor. A well known dissipative p.d.e. is the *complex Ginzburg-Landau equation*, and will be of primary interest in this paper. Additionally, all notation will be introduced to the reader, as needed, and proper definitions will be defined throughout the subsequent sections.

1.1 The *complex Ginzburg-Landau equation*

The *complex Ginzburg-Landau equation* (CGLE) has a detailed history in physics, pattern formation, dynamical systems, etc. as a fundamental p.d.e. modeling the amplitude and phase of instabilities about some critical value or control parameter for many problems in such fields [1],[5]. The derivation of the CGLE will not be detailed here, but one can explore this through papers by Iooss, Mielke and Demay who initially derived the steady CGLE[4]. The two-dimensional CGLE takes the form,

$$\frac{\partial A(\mathbf{x}, t)}{\partial t} = \gamma A(\mathbf{x}, t) + (\lambda + i\alpha)\Delta A(\mathbf{x}, t) - (\kappa + i\beta)A(\mathbf{x}, t)|A(\mathbf{x}, t)|^2, \quad (1)$$

where all the parameters are assumed to be real, $\lambda, \kappa > 0$, $\mathbf{x} \in \mathbb{R}^2$, and $A(\mathbf{x}, t)$ is complex.

One can easily see that setting $\lambda = \kappa = 0$ implies that of the nonlinear Schrödinger equation, and thus the connection of Eq. 1 to laser optics, solitons, fiber optics, etc. seems endless. Dimensionalising Eq. 1,

$$\frac{\partial u(\mathbf{x}, t)}{\partial t} - (1 + i\nu)\Delta u(\mathbf{x}, t) + (1 + i\mu)u(\mathbf{x}, t)|u(\mathbf{x}, t)|^2 - Ru(\mathbf{x}, t) = 0, \quad (2)$$

where $\mathbf{x} \in [0, 1] \times [0, 1]$, $i = \sqrt{-1}$, R takes on the role of a bifurcation parameter, $\nu, \mu \in \mathbb{R}$, and $u(\mathbf{x}, t)$ is again complex¹.

In consideration of the numerous applications of the CGLE it becomes pivotal in accurate understanding of its fundamental characteristics and properties. One of the prominent features of the CGLE is the existence of a global attractor, which is a primary result of the presence of the laplacian operator. In this particular instance the infinite-dimensional system behaves among only a finite number of degrees of freedom. The existence of a global attractor for the CGLE will be explored in detail in **Sec. 3**. Systems that possess an attractor are under the investigation of finding appropriate bounds for the dimension of the attractor. Finite dimensionality of an attractor is guaranteed if one can show that the upper Hausdorff dimension is of at least finite. This will be addressed informally in **Sec. 4**.

2 Theoretical introduction to absorbing sets and attractors

As mentioned before, it is critical to know the fundamental steps in proving a particular model has such wonderful properties. The goal of this section is to do just that.

The layout of this section is as follows. The first subsection will give a brief introduction to semigroups, with the subsequent notation. Following the introduction to semigroups I will define absorbing sets. Extending this definition leads to the introduction of attractors. In addition I will detail some elementary ordinary differential equation (o.d.e.) examples in hopes to grasp particular concepts along the way.

Following these subsections I will state the theorem for existence of a global attractors for a p.d.e. In **Sec. 3** this theorem will be applied to the CGLE².

2.1 Brief introduction to semigroups

We shall assume all of the operators are acting on a Hilbert space in our discussion of semigroups. A semigroup, $S(t)$, contains a family of operators that describe the evolution of a particular system, $u(t)$, with a given initial condition, $u(0)$. In functional notation

$$S(t) : u(0) \rightarrow u(t).$$

In particular, one can label a fixed point of a semigroup if the following condition is satisfied,

$$S(t)u(0) = u(0), \quad \forall t \geq 0.$$

Now consider a collection of fixed points. Call this set \mathbf{X} . Therefore for every element $x \in \mathbf{X}$, x is a fixed point. More importantly, for every element $x \in \mathbf{X}$, $S(t)x \in \mathbf{X}$. Due to this property the set \mathbf{X} is denoted as a functional invariant set for $S(t)$. The formal definition follows.

Definition 1 *A set \mathbf{X} which is a subset of a Hilbert space is a functional invariant set for the semigroup $S(t)$ if*

$$S(t)\mathbf{X} = \mathbf{X}, \quad \forall t \geq 0. \quad \blacksquare$$

¹Changing for $A(\mathbf{x}, t)$ to $u(\mathbf{x}, t)$ is to clearly show the distinction between the dimensional and dimensionless form.

²As promised!

If one has some general sense of what an attractor may be then this definition should make intuitive sense. One should expect that all orbits on the attractor, \mathcal{A} , shall remain on \mathcal{A} when an element of \mathcal{A} is acted on by the associated semigroup. Consequently this will be a necessary condition in the coming definition for an attractor.

2.2 Absorbing sets and global attractors

Loosely speaking, one may think back to ordinary differential equations to think of relatively simple concrete examples of attractors. Along these lines one can categorize attractors into roughly four groups to be named in the following [7].

The first and simplest of these groups are *point* attractors, or systems that have a stable equilibrium point(s). As an example, consider the following system:

$$\dot{y} + 2y + 2 = 0, \tag{3}$$

with prescribed initial conditions. The solution to Eq. 3 can be easily shown to be

$$y(t) = e^{-t} (\rho_1 \cos(t) + \rho_2 \sin(t)), \quad \rho_1, \rho_2 \in \mathbb{R}.$$

In phase space, i.e. \dot{y} vs. y , this system exhibits a stable spiral about the origin. The origin is denoted as a point attractor.

A simple variation of point attractors is to consider a system that has a "line" of fixed points. The word line is meant to capture general 1-d curves of fixed points, which may or may not be periodic. Consider another simple example in radial coordinates

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 1. \tag{4}$$

with prescribed initial conditions. One can show that all orbits approach the unit circle, $r = 1$, in the r, θ plane. This form of a limit cycle is called a *line* attractor. Although this particular example represented a perfect circle this is not always the case! For instance, the **van der Pol equation** can be shown to have a skewed limit cycle that is not circular by any means [6].

From these two examples it is clear that another type of an attractor refers to quasi-periodic behavior, i.e. *surface* attractors [7]. A simple example is to visualize a torus, where all the orbits inside and outside of the torus are attracted to the surface of the torus. The fourth classification of attractors is what is called *strange* attractors, possibly having fractal dimension, and will not be addressed in this paper.

With the necessary semigroup notation introduced in **Sec. 2.1** we can now begin to generalize the principles that define an attractor. Looking over the previous two simple o.d.e. examples one notices two fundamental features. The first being that $\forall x$ on the attractor, \mathcal{A} , $S(t)x$ will remain on the attractor. This implies the existence of functionally invariant set. The second fact is that **any** arbitrary x , in our space, $S(t)x$ will approach \mathcal{A} , i.e. any collection of initial conditions in our space winds up on a strict subspace, \mathcal{A} . It is worth noting that although in these examples any arbitrary x approached \mathcal{A} this is not usually the case, and rather there will exist a basin of attraction of \mathcal{A} such that if x is in this basin then $S(t)x$ will approach \mathcal{A} . In either case this implies that of an absorbing set. This concept will now be formally defined.

Definition 2 Let $B \subset \mathcal{H}$, where \mathcal{H} is a Hilbert space. Let U be an open set containing B . We say that B is absorbing in U if the orbit of any bounded set of U enters into B after a certain time (which may depend on the set). ■

In some sense an attractor is an extension of an absorbing set such that the necessary properties of an attractor implies that of an absorbing set, but not necessarily vice versa. The subtlety will be discussed after the definition is posed.

Definition 3 Let $\mathcal{A} \subset \mathcal{H}$, where \mathcal{H} is a Hilbert space, then \mathcal{A} is called an attractor in \mathcal{H} upon satisfying

1. \mathcal{A} is an invariant set.
2. \mathcal{A} attracts an open set of initial conditions. Therefore \exists an open U containing \mathcal{A} such that if $u_0 \in U$ then $S(t)u_0$ converges to \mathcal{A} as $t \rightarrow \infty$, i.e.

$$\mathbf{dist}(S(t)u_0, \mathcal{A}) \rightarrow 0, \quad t \rightarrow \infty.$$

The largest U satisfying this condition is called the basin of attraction for \mathcal{A} . One says that if the basin of attraction is the whole space (which in this case is \mathcal{H}), then \mathcal{A} is called a global or universal attractor [1], [6].

3. \mathcal{A} is minimal, i.e. there is no proper subset of \mathcal{A} that satisfies (1) and (2) [6]. ■

The subtlety lies in that an attractor is an invariant set which doesn't necessarily hold for an absorbing set. One can find counter examples showing that an absorbing set, B , is not negatively invariant, i.e. $B \not\subseteq S(t)B, \forall t \geq 0$ [5].

2.2.1 Existence of an attractor

Although **Def. 3** makes intuitive sense it is not seem plausible to be able to find that such a set even exists on a particular space associated with a semigroup. Therefore it is beneficial to quantify the necessary conditions that one can show exists for a particular semigroup, in which implies the existence of an attractor. To accomplish this task I will begin outlining the four necessary conditions that implies the existence of an attractor. First I will state two necessary definitions: compactness and the ω -limit set.

Recall the standard definition of compactness.

Definition 4 Let (M, d) be a metric space. A subset $\mathbf{A} \subset M$ is said to be **compact**, if every sequence $\{x_n\} \subset \mathbf{A}$ has a convergence subsequence, $\{x_{n_j}\}$, whose limit is also in \mathbf{A} . Furthermore \mathbf{A} is said to be **relatively compact** if $\bar{\mathbf{A}}$ is **compact**, where the $\bar{\mathbf{A}}$ denotes the closure of \mathbf{A} [2]. ■

In dealing with compact subsets of a metric space, which in this setting is defined on a Hilbert space, one should recall two important features. A compact subset of a metric space is bounded **and** closed.

The other necessary definition is of ω -limits.

Definition 5 Let $\mathcal{A} \subset \mathcal{H}$. Let $S(t)$ be a semigroup on \mathcal{H} . The ω -limit set is defined as,

$$\omega(\mathcal{A}) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)\mathcal{A}},$$

where the over-bar denotes the closure [5], [3]. ■

Let's investigate the meaning of the previous definition. Consider the case where $\omega(\mathcal{A})$ is a single point, \mathbf{x} . Now take a collection of initial conditions which is a subset of our Hilbert space. Call this collection, ironically, \mathcal{A} . Then the ω -limit set says that each individual orbit of $S(t)\mathcal{A}$ after a certain amount of time, depending on each orbit, will be arbitrarily close to \mathbf{x} . One can then expand this definition to understand what it means if the ω -limit set is more than just a point. In that particular case each individual orbit will be arbitrarily close to some point \mathbf{x} after a certain amount of time. Then as more time passes that same orbit will be arbitrarily close to another point in the ω -limit set, and so forth.

Now consider one is given an element \mathbf{x} and is asked to show this is in the ω -limit set of \mathcal{A} . To accomplish this one can show that $\mathbf{x} \in \omega(\mathcal{A})$ if and only if there exists a sequence of elements $\mathbf{x}_n \in \mathcal{A}$ such that $S(t_n)\mathbf{x}_n \rightarrow \mathbf{x}$ [5].

With both of these definitions I will state the theorem³ for the existence of an attractor on a given semigroup with the necessary properties.

³Finally!

Theorem 1 We assume that \mathcal{H} is a Hilbert space. Let B be a bounded subset of \mathcal{H} . Let U be an open subset of \mathcal{H} containing B . Assume that the operators $S(t)$ are given such that the following conditions are satisfied.

1. There exists a functionally invariant set for $S(t)$.
2. $\exists B \subset U$ such that B is absorbing in U on $S(t)$.
3. For all bounded sets \tilde{B} , $\exists t_0$ which may depend on \tilde{B} such that,

$$\bigcup_{t \geq t_0} S(t)\tilde{B}$$

is relatively compact in H .

Then the ω -limit of B (defined as $\mathcal{A} = \omega(B)$), is a compact attractor which attracts the bounded sets of U . It is the maximal bounded attractor in U [5]. ■

Before I begin the proof of this theorem I will state a much needed lemma.

Lemma 1 Suppose for some subset $\mathcal{A} \subset \mathcal{H}$, $\mathcal{A} \neq \emptyset$, and for some $t_0 > 0$ the set $\bigcup_{t \geq t_0} S(t)\mathcal{A}$ is relatively compact in \mathcal{H} . Then $\omega(\mathcal{A})$ is nonempty, compact, and invariant [5]. ■

With this lemma I will begin the proof of **Thm. 1**.

Proof: By the third assumption we can immediately apply the lemma, and therefore we know the $\omega(B)$ is a nonempty, compact, invariant set. Define $\mathcal{A} = \omega(B)$. Therefore, recalling **Def. 3**, all one needs to do to prove that \mathcal{A} is an attractor in U is to show that it attracts all the bounded sets of U , i.e. $\mathbf{dist}(S(t)u_0, \mathcal{A}) \rightarrow 0$, $t \rightarrow \infty$, for all bounded $u_0 \in U$. A proof by contradiction follows.

Suppose \mathcal{A} does not attract all the bounded sets of U . Then there exists at least one u_0 that is **not** attracted by \mathcal{A} . Denote B_0 as the collection of bounded $u_0 \in U$ that are **not** attracted by \mathcal{A} . Then there exists a $\delta > 0$ such that,

$$\mathbf{dist}(S(t)B_0, \mathcal{A}) \geq \delta > 0, \forall t \geq 0.$$

Additionally there is a sequence $\{b_n\}$ such that for all n , $b_n \in B_0$. So

$$\mathbf{dist}(S(t_n)b_n, \mathcal{A}) \geq \delta > \frac{\delta}{2} > 0.$$

By the second assumption, $\exists B$ such that B is absorbing in U . Therefore B must absorb all the bounded sets of B_0 as $B_0 \subset U$. Additionally, since $b_n \in B_0$, then B absorbs b_n for n sufficiently large, $t_n \geq \tilde{t}$, where \tilde{t} represents the time that all orbits in B_0 is absorbed by B . Therefore $S(\tilde{t})b_n \in B$. Since $\bigcup_{t \geq \tilde{t}} S(t)B$ is relatively compact then so is $S(\tilde{t})b_n$. Therefore there exists at least one cluster point β from a convergent subsequence in $S(\tilde{t})b_n$. Let $\{S(t_{n_k})b_{n_k}\}$ denote this subsequence which converges to β , i.e

$$\beta = \lim_{n_k \rightarrow \infty} S(t_{n_k})b_{n_k},$$

and using one of the fundamental semigroup properties,

$$\beta = \lim_{n_k \rightarrow \infty} S(t_{n_k} - \tilde{t})S(\tilde{t})b_{n_k}.$$

Thus β belongs to the $\omega(B)$, i.e. \mathcal{A} . This process can be repeated for every element of B_0 . Therefore \mathcal{A} attracts B_0 , contradicting our assumption. ■

Although there exists alterations for the theorem and the proof [1], [5], I will leave such alterations for the reader to explore⁴. In the next section the theorem of existence of an attractor will be applied to the CGLE.

⁴Translation: to struggle over!

3 Application to the *complex Ginzburg-Landau equation*

I will restate for the reader the dimensionless CGLE,

$$\frac{\partial u}{\partial t} - (1 + i\nu)\Delta u + (1 + i\mu)u|u|^2 - Ru = 0,$$

with Dirichlet, Neumann, or periodic boundary conditions on $\Omega = [0, 1] \times [0, 1]$. The initial condition takes on the prescribed value of $u(\mathbf{x}, 0) = u_0(\mathbf{x})$, $\mathbf{x} \in \Omega$. Furthermore, Eq. 2 lives on a Hilbert space of $L^2(\mathbb{C}, \Omega)$ functions and the solutions are on a subspace, \mathbf{V} , of \mathcal{H} . Depending on the three types of boundary conditions, listed above, \mathbf{V} takes on three possible forms,

$$\begin{aligned} \mathbf{V} &= \mathcal{H}_0^1(\mathbb{C}, \Omega) = \left\{ u \in \mathcal{H} \mid \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in \mathcal{H}, u = 0 \text{ on } \partial\Omega \right\}, \\ \mathbf{V} &= \mathcal{H}^1(\mathbb{C}, \Omega) = \left\{ u \in \mathcal{H} \mid \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in \mathcal{H}, \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}, \\ \mathbf{V} &= \mathcal{H}_{per}^1(\mathbb{C}, \Omega) = \left\{ u \in \mathcal{H} \mid \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in \mathcal{H}, u \text{ is periodic on } \partial\Omega \right\}. \end{aligned}$$

For the time being it will be assumed that the spaces are to be taken over the complex plane in the given domain Ω . Something to additionally note is that the initial condition, $u_0(\mathbf{x})$, is only assumed to be in \mathcal{H} and not necessarily in \mathbf{V} . For the CGLE there exists a unique solution, $u(\mathbf{x}) \in \mathbf{V}$, which can be broken into two cases for a given $u_0 \in \mathcal{H}$ [5]:

1. Given $u_0 \in \mathcal{H}$ then for the functional evolution equation (CGLE) there exists a unique solution up to time $T < \infty$ such that,

$$u \in \{\text{continuous functions in } \mathcal{H}\} \cap \mathbf{L}^2(V).$$

2. Given $u_0 \in \mathbf{V} \subsetneq \mathcal{H}$ then for the functional evolution equation (CGLE) there exists a unique solution up to time $T < \infty$ such that,

$$u \in \{\text{continuous functions in } \mathbf{V}\} \cap \mathbf{L}^2(V \cap \mathcal{H}^2(\Omega)).$$

With this existence and uniqueness statement one can define a semigroup, $S(t)$ [5], i.e. $S(t) : u_0(x) \rightarrow u(x, t)$. Thus the theorem for the existence of an attractor for the CGLE will now be stated and proven.

Theorem 2 *The complex Ginzburg-Landau equation with either Dirichlet, Neumann, or periodic boundary conditions, possesses a global attractor \mathcal{A} in $L^2(\mathbb{C}, \Omega)$.* ■

Proof: I will outline the goals of the proof in words. First, we know that for all general initial conditions in our complex Hilbert space our unique solution is in \mathbf{V} . Therefore we shall show that for all $u_0(\mathbf{x})$ the associated trajectory $u(\mathbf{x}, t)$ is bounded in the associated Sobolev⁵ norm on \mathbf{V} . To accomplish this task we shall obtain bounds for $\|u\|_{L^2}^2$ and $\|\nabla u\|_{L^2}^2$, which may depend on time. These bounds can be viewed as a ball, B_t , in \mathbf{V} such that as $t \rightarrow \infty$ the ball, B_t , becomes minimal. Call this ball B_0 . Therefore B_0 is a functional invariant set, which is absorbing in \mathcal{H} as all initial conditions, after a large enough time, enter or approach B_0 . Additionally this will lead to satisfying the second part of **Thm. 1**, i.e. relatively compactness in \mathcal{H} , as it is given that the injection or mapping of \mathbf{V} in \mathcal{H} is compact [5]. Then **Thm. 1** will be applied to complete the proof. Before I begin the proof I will state two important lemmas that will be continuously used in the following proof. The proof of the lemmas can be found quite easily in an ordinary differential equations book.

⁵Recall that the Sobolev norm for some $\phi \in \mathbf{V}$ is defined as $(\|\phi\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2)^{1/2}$.

Lemma 2 The Gronwall Lemma: Let $g, h, y, \frac{d}{dt}y$ be four locally integrable functions on (t_0, ∞) that satisfy,

$$\frac{d}{dt}y \leq gy + h, \quad \forall t \geq t_0,$$

then the $y(t)$ satisfies the following inequality,

$$y(t) \leq y(t_0) \exp\left(-\int_{t_0}^t g(\tau) d\tau\right) + \int_{t_0}^t h(s) \exp\left(-\int_t^s g(\tau) d\tau\right) ds. \quad \blacksquare$$

The uniform Gronwall Lemma: Let everything be the same as above for $g, h, y, \frac{d}{dt}y$. Let a_1, a_2, a_3 be three positive constants such that for $r > 0$,

$$\int_t^{t+r} g(s) ds \leq a_1, \quad \int_t^{t+r} h(s) ds \leq a_2, \quad \int_t^{t+r} y(s) ds \leq a_3, \quad \forall t \geq t_0.$$

Then the following inequality holds,

$$y(t+r) \leq \left(\frac{a_3}{r} + a_2\right) \exp(a_1), \quad \forall t \geq t_0. \quad \blacksquare$$

The proof now begins. To accomplish the task of showing the existence of an absorbing set, B_0 in \mathcal{H} , we shall show first that the L^2 norm of u is bounded. Therefore we shall obtain the energy equation for Eq. 2, which consequently is one dimensional. Multiplying Eq. 2 by u^* and integrating over Ω

$$\int_{\Omega} u^* u_t d\mathbf{x} - (1 + i\nu) \int_{\Omega} u^* \Delta u d\mathbf{x} + (1 + i\mu) \int_{\Omega} |u|^4 d\mathbf{x} - R \int_{\Omega} |u|^2 d\mathbf{x} = 0.$$

Therefore one can examine solely the real part⁶

$$\int_{\Omega} \operatorname{Re}(u^* u_t) d\mathbf{x} - \int_{\Omega} u^* \Delta u d\mathbf{x} + \int_{\Omega} |u|^4 d\mathbf{x} - R \int_{\Omega} |u|^2 d\mathbf{x} = 0.$$

Applying Green's formula and rewriting in norm notation

$$\int_{\Omega} \operatorname{Re}(u^* u_t) d\mathbf{x} - \int_{\partial\Omega} u^* \frac{\partial u}{\partial n} ds + \int_{\Omega} \nabla u^* \nabla u d\mathbf{x} + \|u\|_{L^4}^4 - R \|u\|_{L^2}^2 = 0.$$

Using the associated boundary conditions

$$\int_{\Omega} \operatorname{Re}(u^* u_t) d\mathbf{x} + \|\nabla u\|_{L^2}^2 + \|u\|_{L^4}^4 - R \|u\|_{L^2}^2 = 0.$$

A usefal fact to be used is $\frac{1}{2} \left(\frac{\partial}{\partial t}(|u|^2)\right) = \operatorname{Re}(u^* u_t)$. With this last equality and noting the integration in \mathbf{x} is definite one obtains the following o.d.e.

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|u\|_{L^4}^4 - R \|u\|_{L^2}^2 = 0. \quad (5)$$

Now depending on the value of the bifurcation parameter, R , there are three cases that need to be addressed. For $R < 0$ it is simple to see this leads to our solution converging to the origin as Eq. 5 can be written as,

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + |R| \|u\|_{L^2}^2 \leq 0.$$

⁶Recall given $z = 0$ if and only if $\operatorname{Re}(z) = \operatorname{Im}(z) = 0$

Now applying the usual Gronwall lemma

$$\|u\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 \exp(-|R|t),$$

which clearly goes to zero as $t \rightarrow \infty$. For $R = 0$ the solution still has trivial dynamics, but the proof isn't as straightforward⁷. Using Hölder's inequality,

$$\|u\|_{L^2}^2 \leq |\Omega|^{\frac{1}{2}} \|u\|_{L^4}^2.$$

Therefore one can rewrite Eq. 5 as

$$\frac{d}{dt} \|u\|_{L^2}^2 + \frac{2}{|\Omega|} \|u\|_{L^2}^4 \leq 0.$$

Or simply as,

$$\frac{\dot{y}}{y^2} \leq -\frac{2}{|\Omega|}, \quad \text{where } y = \|u\|_{L^2}^2.$$

Thus by integration the following inequality holds,

$$\frac{1}{\|u_0\|_{L^2}^2} + \frac{2}{|\Omega|} t \leq \frac{1}{\|u\|_{L^2}^2}.$$

Therefore as $t \rightarrow \infty$, $\|u\|_{L^2}^2$ must approach zero.

The remaining case, $R > 0$, will now be the primary focus. I will now state another needed lemma.

Lemma 3 *Given the polynomial in s , $p(s) = \frac{1}{2}s^4 - 2Rs^2$ where $R > 0$ then*

$$\frac{1}{2}s^4 - 2Rs^2 \geq -2R^2.$$

Proof: *Since $\frac{1}{2}(s^2 - 2R)^2 \geq 0$ then $\frac{1}{2}s^4 - 2Rs^2 + 2R^2 \geq 0$ and thus $p(s) \geq -2R^2$.* ■

Eq. 5 can be rewritten as

$$\frac{d}{dt} \|u\|_{L^2}^2 + 2\|\nabla u\|_{L^2}^2 + 2\|u\|_{L^4}^4 + 2R\|u\|_{L^2}^2 = 4R\|u\|_{L^2}^2. \quad (6)$$

From the lemma one can deduce, by letting $s = |u|$ and integrating over Ω , that

$$\frac{1}{2}\|u\|_{L^4}^4 - 2R\|u\|_{L^2}^2 \geq -2R^2|\Omega|.$$

The above inequality can be written in a more useful form,

$$4R\|u\|_{L^2}^2 \leq 4R^2|\Omega| + \|u\|_{L^4}^4.$$

Applying this to Eq. 6,

$$\frac{d}{dt} \|u\|_{L^2}^2 + 2\|\nabla u\|_{L^2}^2 + \|u\|_{L^4}^4 + 2R\|u\|_{L^2}^2 \leq 4R^2|\Omega|. \quad (7)$$

Therefore it is clear that

⁷At least for myself it wasn't!

$$\frac{d}{dt} \|u\|_{L^2}^2 \leq -2R \|u\|_{L^2}^2 + 4R^2 |\Omega|.$$

Now we are set to use the Gronwall lemma, with $y = \|u\|_{L^2}^2$, $g = -2R$, and $h = 4R^2 |\Omega|$. Applying the Gronwall lemma, and using the fact $\exp(-2Rt) \leq \exp(-Rt)$ for $t \geq 0$, one obtains an upper bound for $\|u\|_{L^2}^2$,

$$\|u\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 \exp(-Rt) + 2R |\Omega| (1 - \exp(-Rt)). \quad (8)$$

Taking the limit as $t \rightarrow \infty$,

$$\limsup_{t \rightarrow \infty} \|u\|_{L^2}^2 \leq 2R |\Omega|.$$

Therefore the L^2 norm of u has been shown to be bounded and we are left with showing that the L^2 norm of ∇u is bounded as well.

I will define $B_{\mathcal{H}}$ as the ball in \mathcal{H} with a finite radius $\rho_{\mathcal{H}} > \rho_0 = \sqrt{2R|\Omega|}$. Clearly $B_{\mathcal{H}}$ is absorbing in \mathcal{H} , as all orbits end up in $B_{\mathcal{H}}$ for sufficiently large t . Further analysis can be done to find the value of t that all the orbits belong to $B_{\mathcal{H}}$, i.e. $S(t)B \subset B_{\mathcal{H}}$ for $t \geq t_0$. Let B_r be a bounded subset of \mathcal{H} and let $B \subset B_r$. We shall find a time, t_0 , such that,

$$\|u\|_{L^2}^2 \leq \rho_H^2. \quad (9)$$

Using Eq. 8 the above inequality is satisfied at t_0 if

$$\rho_H^2 = \|u_0\|_{L^2}^2 \exp(-Rt_0) + \rho_0^2 (1 - \exp(-Rt_0)) \leq r^2 \exp(-Rt_0) + \rho_0^2.$$

Taking the natural logarithm and solving for t_0 one obtains

$$t_0 = \frac{1}{R} \ln \left(\frac{r^2}{\rho_{\mathcal{H}}^2 - \rho_0^2} \right). \quad (10)$$

A useful inequality which will be used later can be derived easily from Eq. 10. Integrating Eq. 7 between t and $t+r$ for $r > 0$ and $t \geq t_0$,

$$\int_t^{t+r} \left(\frac{d}{dt} \|u\|_{L^2}^2 + 2 \|\nabla u\|_{L^2}^2 + \|u\|_{L^4}^4 + 2R \|u\|_{L^2}^2 \right) dt \leq 4rR^2 |\Omega|.$$

Since $t \geq t_0$ then $\|u\|_{L^2}^2 \leq \rho_{\mathcal{H}}^2$. So

$$\int_t^{t+r} (2 \|\nabla u\|_{L^2}^2 + \|u\|_{L^4}^4 + 2R \|u\|_{L^2}^2) dt \leq \rho_{\mathcal{H}}^2 + 4rR^2 |\Omega|. \quad (11)$$

Now we shall focus our attention on bounding ∇u . The goal is to show that $\forall u \in B_{\mathcal{H}}$ the L^2 norm of the gradient of u is also bounded. In other words, we shall show that there exists $B_V \subset \mathbf{V}$ such that $B_{\mathcal{H}} \subset B_V$ and all the orbits are absorbed in the sense of the L^2 norm of their associated gradient. These two conclusions will prove the existence of an absorbing set for the associated semigroup.

To accomplish this task, we shall obtain another energy equation by multiplying Eq. 2 by $-\Delta u^*$ and integrating over Ω ,

$$\int_{\Omega} -\Delta u^* u_t dx + (1 + i\nu) \int_{\Omega} |\Delta u|^2 dx - (1 + i\mu) \int_{\Omega} \Delta u^* |u|^2 u dx + R \int_{\Omega} \Delta u^* u dx = 0.$$

Similarly, as before, one can easily show by Green's formula that

$$\int_{\Omega} \nabla u^* \nabla u_t d\mathbf{x} + (1 + i\nu) \|\Delta u\|_{L^2}^2 - (1 + i\mu) \int_{\Omega} \nabla(u|u|^2) \cdot \nabla u^* d\mathbf{x} - R \|\nabla u\|_{L^2}^2 = 0.$$

With the identity $\frac{1}{2} \left(\frac{\partial}{\partial t} (|u|^2) \right) = \text{Re}(u^* u_t)$, noticing the integration is definite, and taking the real part of the resulting equation one obtains

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 - R \|\nabla u\|_{L^2}^2 = \text{Re} \left((1 + i\mu) \int_{\Omega} \nabla(u|u|^2) \cdot \nabla u^* d\mathbf{x} \right).$$

On further examination of the resulting integrand of the remaining integral one can show

$$\begin{aligned} \nabla(u|u|^2) \cdot \nabla u^* &= (|u|^2 \nabla u + u \nabla |u|^2) \cdot \nabla u^* \\ &= |u|^2 |\nabla u|^2 + u (u^* \nabla u + u \nabla u^*) \cdot \nabla u^* \\ &= 2|u|^2 |\nabla u|^2 + u^2 |\nabla u|^2 \leq 3|u|^2 |\nabla u|^2 \end{aligned}$$

Therefore on using this result one arrives at the following inequality

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 - R \|\nabla u\|_{L^2}^2 &\leq 3(1 + \mu^2)^{1/2} \int_{\Omega} |u|^2 |\nabla u|^2 d\mathbf{x} \\ &\leq 3(1 + \mu^2)^{1/2} \|u\|_{L^4}^2 \|\nabla u\|_{L^4}^2, \end{aligned} \quad (12)$$

where the last inequality is found through the Cauchy Schwartz inequality.

One can bound $\|\nabla u\|_{L^4}^2$ by recalling a few useful facts. In particular, if one is given two Hilbert spaces and assuming a few necessary conditions, on both spaces, interpolation theory yields a way of developing an intermediate space between these two spaces [5]. In our case one can think of the two spaces as the complex version of $L^2(\Omega)$ and the associated Sobolev norm on \mathbf{V} . The intermediate space would be the complex version of $L^4(\Omega)$. This useful fact enables us to bound an element in the intermediate space by the associated norms of that element in the other two spaces to within an arbitrary constant. Applying this fact

$$\frac{d}{dt} \|\nabla u\|_{L^4}^2 \leq c_1^2 \|\nabla u\|_{L^2} (\|\Delta u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)^{1/2}.$$

As the Sobolev norm of $\|\nabla u\|_{L^2}^2$ is equivalent to the natural Sobolev norm of $H^2(\Omega)$,

$$\frac{d}{dt} \|\nabla u\|_{L^4}^2 \leq c_2^2 \|\nabla u\|_{L^2} (\|\Delta u\|_{L^2}^2 + \|u\|_{L^2}^2)^{1/2},$$

where c_2^2 is a constant from the equivalence of the norms multiplying c_1^2 . Replacing this inequality into Eq. 12,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 - R \|\nabla u\|_{L^2}^2 &\leq 3c_2^2 (1 + \mu^2)^{1/2} \|u\|_{L^4}^2 \|\nabla u\|_{L^2} (\|\Delta u\|_{L^2}^2 + \|u\|_{L^2}^2)^{1/2} \\ &\leq \frac{9}{2} c_2^4 (1 + \mu^2) \|u\|_{L^4}^4 \|\nabla u\|_{L^2}^2 + \frac{1}{2} \|\Delta u\|_{L^2}^2 + \frac{1}{2} \|u\|_{L^2}^2, \end{aligned}$$

where the last inequality is found using the fact that $2ab \leq a^2 + b^2$. From this inequality it is easily seen that

$$\frac{d}{dt} \|\nabla u\|_{L^2}^2 \leq 2(R + c_3 \|u\|_{L^4}^4) \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2, \quad (13)$$

$$c_3 = \frac{9}{2} c_2^4 (1 + \mu^2). \quad (14)$$

At this point Eq. 13 is in perfect position to apply the uniform Gronwall lemma with,

$$y = \|\nabla u\|_{L^2}^2, \quad g = 2(R + c_3\|u\|_{L^4}^4), \quad h = \|u\|_{L^2}^2.$$

As noted in the lemma, the integral from t to $t+r$ of y , g , h , must be bounded by three positive constants a_1 , a_2 , a_3 . This is easily seen using Eq. 11 for the bounds for y and g , and Eq. 9 for h , assuming $t \geq t_0$. Thus,

$$\begin{aligned} \int_t^{t+r} g \, ds &= 2Rr + \int_t^{t+r} 2c_3\|u\|_{L^4}^4 \, ds \leq 2Rr + 2c_3(\rho_H^2 + 4rR^2|\Omega|) = a_1, \\ \int_t^{t+r} h \, ds &= \int_t^{t+r} \|u\|_{L^2}^2 \, ds \leq \rho_H^2 = a_2, \\ \int_t^{t+r} y \, ds &= \int_t^{t+r} \|\nabla u\|_{L^2}^2 \, ds \leq \frac{\rho_H^2}{2} + 2rR|\Omega| = a_3. \end{aligned}$$

Applying the lemma,

$$\|\nabla u\|_{L^2}^2 \leq \left(\frac{a_3}{r} + a_2\right) \exp(a_1), \quad \forall t \geq t_0. \quad (15)$$

Looking closer at Eq. 15 one can denote this as another ball, B_1 , where $B_{\mathcal{H}} \subsetneq B_1$. Thus after some t_0 both the $\|\nabla u\|_{L^2}^2$ and $\|u\|_{L^2}^2$ will be in $B_{\mathcal{H}}$. Thus $B_{\mathcal{H}}$ is a global absorbing set in \mathcal{H} for the semigroup $S(t)$, and is consequently minimal. In addition for any $B \subset \mathcal{H}$, $S(t)B \subset B_{\mathcal{H}}$ after some time, t_0 . Thus $S(t)B$ is also a strict subset of \mathbf{V} for $t \geq t_0$. Using the fact that the injection of \mathbf{V} in \mathcal{H} is compact, the closure of $\bigcup_{t>t_0} S(t)B$, i.e. the union of all possible orbits after some time t_0 , must be compact as well. Thus all parts of **Thm. 1** apply and the proof is complete. \blacksquare

4 Bounding the dimensionality of an attractor

The existence of a global attractor leads to the conclusion that our infinite-dimensional system will be controlled by a subspace of \mathcal{H} . Notice that this in no way guarantees that the attractor is finite-dimensional! It is still left to show that the Hausdorff dimension of the attractor is indeed finite-dimensional. Unfortunately finding the exact dimension is often not tangible and thus finding a priori estimates is the next best approach.

The most standard approach is bounding the dimension from below. One way to accomplish this is by performing a linear stability analysis about a known equilibrium solution u_{eq} . Loosely speaking, then one may perturb u_{eq} and analyze the growth of the perturbation. This allows one to bound the lower dimension of the attractor by the number or collection of unstable wavenumbers that exhibit growth. This will not be the primary focus of this section, but an outline of this analysis for the CGLE is given in [1]. Instead the focus will be shifted on bounding the dimensionality of an attractor from above and an informal discussion follows.

To bound the dimension from above we consider two projection operators, \mathbf{P} and \mathbf{Q} , which will decompose the infinite-dimensional space in two disjoint sets. The projection operator, \mathbf{P} , which (hopefully) will be shown to be finite, maps a given solution to a space \mathbf{X} and \mathbf{Q} maps a given solution to $\mathcal{H} \setminus \mathbf{X}$. The goal is twofold. First one needs to show the cone condition is satisfied, i.e.

$$\|\mathbf{Q}a(t)\|_2 \leq \|\mathbf{P}a(t)\|_2, \quad (16)$$

where $a(t)$ is the difference between two solutions $A_1(t)$ and $A_2(t)$ [1]. The second part which should be intuitively obvious, is the need to show \mathbf{P} is a finite N-dimensional projection. If this can be shown, referring back to Eq. 16, one sees that the energy in the system will be dominated by a finite number of fourier modes.

For dissipative systems, as in the case of the CGLE, the finite dimensionality of \mathbf{P} is guaranteed by the spectrum of the laplacian or the presence of higher order dissipative operators [1], [5]. This is one of the nice features of dissipative systems. Thus the remaining question is to find the smallest value of N ($N < \infty$) for which Eq. 16 is satisfied. Due to the finite dimensionality of \mathbf{P} one can write down a one-to-one function, Φ , to accomplish this task of finding N ,

$$\Phi : \mathbf{X} \subset \mathbf{P}\mathcal{H} \mapsto \mathbf{Q}\mathcal{H} \quad [1].$$

In particular, one is interested in the set $\mathbf{X} = \mathbf{P}\mathcal{A}$, where \mathcal{A} is the global attractor at hand. Thus for a given initial condition, $A(0, \mathbf{x}) = A_0(\mathbf{x})$, with a solution $A(t, \mathbf{x})$ satisfying the governing infinite-dimensional system, one can rewrite $A(t, \mathbf{x})$ into an N -dimensional expansion and its remainder. More explicitly

$$A(t, \mathbf{x}) = \hat{a}(t, \mathbf{x}) + \Phi[\hat{a}(t, \mathbf{x})],$$

where \hat{a} is an N -dimensional fourier expansion⁸ in $\mathbf{P}\mathcal{A}$ and $\Phi[\hat{a}(t, \mathbf{x})]$ maps \hat{a} into its infinite-dimensional counterpart. Additionally each fourier mode must satisfy the governing p.d.e. with the initial condition $\hat{a} = \mathbf{P}A_0$. This in turn reduces the infinite-dimensional system to a system of ordinary differential equations for the fourier coefficients. This decomposition allows one to bound the upper Hausdorff dimension of the global attractor by $2N + 1$, i.e. the number of differential equations to be solved for each resulting fourier mode [1].

5 Conclusions

As noted before many infinite-dimensional systems may be controlled by a rather small number of degrees of freedom. Although the presence of a global attractor does not guarantee this it does motivate one to explore this possibility. For systems such as the CGLE, the dissipative nature ensures the finite dimensionality of the existing global attractor [3].

Showing the existence of a global attractor is the first step in exploring the finite dimensionality of an infinite-dimensional system. In doing so, as shown with CGLE, one begins to understand the asymptotic behavior of solutions in the configuration space. In particular it is something to note that showing the finite dimensionality of an attractor in some sense means the attractor is not as deeply embedded in the infinite-dimensional configuration space [1]. This not only enhances our knowledge of a particular solution's long term behavior, but also provides the groundwork and validity to explore solutions computationally through variational methods, weighted residuals, and/or spectral methods. Providing the existence of a global attractor can be shown, it becomes ever increasingly important to tightly bound the upper Hausdorff dimension.

⁸This is just the Galerkin approximation to the p.d.e. at hand! Recall that a Galerkin approach one approximates the solution by a finite number of basis functions (eigenfunctions),

$$A(t, \mathbf{x}) = \sum_{i=-N}^N \hat{a}_i(t) \psi_i(\mathbf{x}),$$

where $\psi_i(\mathbf{x})$ are the associated basis functions.

References

- [1] DOERING, C., GIBBON, J., HOLM, D., NICOLAENKO B., "Low-Dimensional behaviour in the complex Ginzburg-Landau equation", *Nonlinearity* **1**, **279-309** (1988).
- [2] FLASCHKA, H., *Principles of Analysis*, **303-304** (1991).
- [3] HALE, J., *Asymptotic Behavior of Dissipative Systems*, AMS, Number 25, **27-28**, **35-39**, **168-170** (1988).
- [4] MELBOURNE, I., "Derivation of the time-dependent Ginzburg-Landau equation on the line", *J. Nonlinear Science* **8**, **1-15** (1998).
- [5] TEMAM, R., *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd. Edition, Springer, NY, **16-27**, **45-48**, **56-58**, **84-88**, **90-91**, **226-234** (1997).
- [6] STROGATZ, S., *Nonlinear Dynamics and Chaos*, Perseus Books, MA, **197-199**, **324-325** (1994).
- [7] WALGRAEF, D., *Spatio-Temporal Pattern Formation, with Examples from Physics, Chemistry, and Materials Science*, Springer, NY, **245-247**, **269** (1997).