1 Groups and dual groups

Fourier analysis relates functions on a group G to functions on a dual group $\hat{G}$. Here are some examples.

Real line and dual real line $G = \mathbf{R}$ and $\hat{G} = \mathbf{R}$. We have the inverse Fourier transform

$$f(x) = \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k) \, \frac{dk}{2\pi} \tag{1}$$

where the Fourier transform is

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) \, dx. \tag{2}$$

Circle and dual integers $G = T_L$ and $\hat{G} = \delta k \, \mathbf{Z}$. Here $\delta k = \frac{2\pi}{L}$. We have the Fourier series

$$f(x) = \sum_{p=-\infty}^{\infty} e^{ip \, \delta k \, x} \hat{f}(p \, \delta k) \, \frac{\delta k}{2\pi}. \tag{3}$$

where the Fourier coefficients are

$$\hat{f}(p \, \delta k) = \int_0^L e^{-ip \, \delta k \, x} f(x) \, dx. \tag{4}$$

Finite cyclic group and dual finite cyclic group $G = \Delta x \, \mathbf{Z}_N$ and $\hat{G} = \delta k \, \mathbf{Z}_N$. Here $\delta k = \frac{2\pi}{N \Delta x}$. We have the discrete inverse Fourier transform

$$f(n\Delta x) = \sum_{p=0}^{N-1} e^{ip \, \delta k \, n \Delta x} \hat{f}(p \, \delta k) \, \frac{\delta k}{2\pi}. \tag{5}$$

where the discrete Fourier transform is

$$\hat{f}(k) = \sum_{n=0}^{N-1} e^{-ip \, \delta k \, n \Delta x} f(n\Delta x) \, \Delta x. \tag{6}$$

2 Poisson summation formula

The setting for the Poisson summation formula is a function f on a group G and its Fourier transform $\hat{f}$ on the dual group $\hat{G}$. However now there is a given subgroup $H \subseteq G$. Furthermore, there is a quotient group $G\mathrm{mod}H$. The dual group of $G\mathrm{mod}H$ is a subgroup of the dual group of G. The Poisson summation related a sum involving f over H with a sum involving $\hat{f}$ over the dual group of $G\mathrm{mod}H$.

Integer subgroup of real line $G = \mathbf{R}$ and $\hat{G} = \mathbf{R}$. The subgroup $H = \Delta x\, \mathbf{Z}$ and the quotient group $G\mathrm{mod}H = T_{\Delta x}$ with dual group $\Delta k\, \mathbf{Z}$, where $\Delta k = \frac{2\pi}{\Delta x}$. We have

$$\sum_{n=-\infty}^{\infty} f(x + n\Delta x) = \sum_{m=-\infty}^{\infty} e^{im\Delta k\, x} \hat{f}(m\Delta k)\, \frac{\Delta k}{2\pi} \tag{7}$$

and also

$$\sum_{m=-\infty}^{\infty} \hat{f}(k + m\, \Delta k) = \sum_{n=-\infty}^{\infty} e^{-ikn\Delta x} f(n\Delta x)\, \Delta x. \tag{8}$$

The last equation says that if we try to compute the Fourier transform of the function $f(x)$ by a discrete sum with spacing Δx, then we get the answer $\hat{f}(k)$ plus an infinite number of alias results $\hat{f}(k + m\Delta k)$ with $m \neq 0$.

Finite cyclic subgroup of circle $G = T_L$ and $\hat{G} = \delta k\, \mathbf{Z}$, where $\delta k = \frac{2\pi}{L}$. The subgroup $H = \Delta x\, \mathbf{Z}_N$, where $N\Delta x = L$. The quotient group $G\mathrm{mod}H = T_{\Delta x}$ with dual group $\Delta k\, \mathbf{Z}$, where $\Delta k = \frac{2\pi}{\Delta x} = N\delta k$. We have

$$\sum_{n=0}^{N-1} f(x + n\Delta x) = \sum_{m=-\infty}^{\infty} e^{im\Delta k\, x} \hat{f}(m\Delta k)\, \frac{\Delta k}{2\pi} \tag{9}$$

and

$$\sum_{m=-\infty}^{\infty} \hat{f}(p\, \delta k + m\Delta k) = \sum_{n=0}^{N-1} e^{-ip\, \delta k\, n\Delta x} f(n\Delta x)\, \Delta x. \tag{10}$$

The last equation says that if we try to compute the Fourier coefficients of the periodic function $f(x)$ by a finite sum with spacing Δx, then we get the answer $\hat{f}(p\, \delta k)$ plus an infinite number of alias results $\hat{f}(p\, \delta k + m\Delta k)$ with $m \neq 0$.