Chapter 1

Fourier series

1.1 Orthonormal families

Let T be the circle parameterized by $[0, 2\pi)$ or by $[-\pi, \pi)$. Let f be a complex function on T that is integrable. The nth Fourier coefficient is

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} f(x)\, dx. \tag{1.1}$$

The goal is to show that f has a representation as a Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}. \tag{1.2}$$

There are two problems. One is to interpret the sense in which the series converges. The second is to show that it actually converges to f.

This is a huge subject. However the simplest and most useful theory is in the context of Hilbert space. Let $L^2(T)$ be the space of all (Borel measurable) functions such that

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2\, dx < \infty. \tag{1.3}$$

Then $L^2(T)$ is a Hilbert space with inner product

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} \overline{f(x)} g(x)\, dx. \tag{1.4}$$

Let

$$\phi_n(x) = \exp(inx). \tag{1.5}$$

Then the ϕ_n form an orthogonal family in $L^2(T)$. Furthermore, we have the identity

$$\|f\|^2 = \sum_{|n| \le N} |c_n|^2 + \|f - \sum_{|n| \le N} c_n \phi_n\|^2. \tag{1.6}$$

In particular, we have Bessel's inequality

$$\|f\|^2 \geq \sum_{|n|\leq N} |c_n|^2. \tag{1.7}$$

This shows that

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty. \tag{1.8}$$

The space of sequences satisfying this identity is called ℓ^2. Thus we have proved the following theorem.

Theorem. If f is in $L^2(T)$, then its sequence of Fourier coefficients is in ℓ^2.

1.2 L^2 convergence

It is easy to see that $\sum_{|n|\leq N} c_n \phi_n$ is a Cauchy sequence in $L^2(T)$ as $N \to \infty$. It follows from the completeness of $L^2(T)$ that it converges in the L^2 sense to a sum g in $L^2(T)$. That is, we have

$$\lim_{N\to\infty} \|g - \sum_{|n|\leq N} c_n \phi_n\|^2 = 0. \tag{1.9}$$

The only remaining thing to show is that $g = f$. This, however, requires some additional ideas.

One way to do it is to take r with $0 < r < 1$ and look at the sum

$$\sum_{n=-\infty}^{\infty} r^{|n|} c_n e^{inx} = \frac{1}{2\pi} \int_0^{2\pi} P_r(x-y) f(y)\, dy. \tag{1.10}$$

Here

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx} = \frac{1-r^2}{1-2r\cos(x)+r^2}. \tag{1.11}$$

The functions $\frac{1}{2\pi} P_r(x)$ have the properties of an approximate delta function. Each such function is positive and has integral 1 over the periodic interval. Furthermore,

$$P_r(x) \leq \frac{1-r^2}{2r(1-\cos(x))}, \tag{1.12}$$

which approaches zero as $r \to 1$ away from points where $\cos(x) = 1$.

Theorem. If f is continuous on T, then

$$f(x) = \lim_{r\uparrow 1} \sum_{n=-\infty}^{\infty} r^{|n|} c_n e^{inx}, \tag{1.13}$$

and the convergence is uniform.

Proof: It is easy to compute that

$$\sum_{n=-\infty}^{\infty} r^{|n|} c_n e^{inx} = \frac{1}{2\pi}\int_0^{2\pi} P_r(x-y) f(y)\, dy = \frac{1}{2\pi}\int_0^{2\pi} P_r(z) f(x-z)\, dz. \quad (1.14)$$

Hence

$$f(x) - \sum_{n=-\infty}^{\infty} r^{|n|} c_n e^{inx} = \frac{1}{2\pi}\int_0^{2\pi} P_r(z)(f(x) - f(x-z))\, dz. \quad (1.15)$$

Thus

$$|f(x) - \sum_{n=-\infty}^{\infty} r^{|n|} c_n e^{inx}| = |\frac{1}{2\pi}\int_0^{2\pi} P_r(z)(f(x)-f(x-z))\, dz| \leq \frac{1}{2\pi}\int_0^{2\pi} P_r(z)|f(x)-f(x-z)|\, dz. \quad (1.16)$$

Since f is continuous on T, its absolute value is bounded by some constant M. Furthermore, since f is continuous on T, it is uniformly continuous on T. Consider arbitrary $\epsilon > 0$. It follows from the uniform continuity that there exists $\delta > 0$ such that for all x and all z with $|z| < \delta$ we have $|f(x) - f(x-z)| < \epsilon/2$. Thus the right hand side is bounded by

$$\frac{1}{2\pi}\int_{|z|<\delta} P_r(z)|f(x)-f(x-z)|\, dz + \frac{1}{2\pi}\int_{|z|\geq\delta} P_r(z)|f(x)-f(x-z)|\, dx \leq \epsilon/2 + \frac{1-r^2}{2r(1-\cos(\delta))} 2M. \quad (1.17)$$

Take r so close to 1 that the second term is also less than $\epsilon/2$. Then the difference that is being estimated is less than ϵ for such r. This proves the result.

Theorem. If f is in $L^2(T)$, then

$$f = \sum_{n=-\infty}^{\infty} c_n \phi_n \quad (1.18)$$

in the sense that

$$\lim_{N\to\infty} \|f - \sum_{|n|\leq N} c_n \phi_n\|^2 = 0. \quad (1.19)$$

Proof: Let f have Fourier coefficients c_n and $g = \sum_n c_n \phi_n$. Let $\epsilon > 0$ be arbitrary. Let h be in $C(T)$ with

$$\|f - h\|_2 < \epsilon/2. \quad (1.20)$$

Suppose h has Fourier coefficients d_n. Then there exists $r < 1$ such that

$$\|h - \sum_n r^n d_n e^{inx}\|_\infty < \epsilon/2. \quad (1.21)$$

It follows that

$$\|h - \sum_n r^n d_n e^{inx}\|_2 < \epsilon/2. \quad (1.22)$$

Thus

$$\|f - \sum_n r^n d_n e^{inx}\|_2 < \epsilon. \tag{1.23}$$

However since g is the orthogonal projection of f onto the span of the ϕ_n, it is also the best approximation of f in the span of the ϕ_n. Thus

$$\|f - g\|_2 < \epsilon. \tag{1.24}$$

Since $\epsilon > 0$ is arbitrary, we can only conclude that $f = g$.

Theorem. If f is in $L^2(T)$, then

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{1.25}$$

1.3 Absolute convergence

Define the function spaces

$$C(T) \subset L^\infty(T) \subset L^2(T) \subset L^1(T). \tag{1.26}$$

The norms $\|f\|_\infty$ on the first two spaces are the same, the smallest number M such that $|f(x)| \le M$ (with the possible exception of a set of x of measure zero). The space $C(T)$ consists of continuous functions; the space $L^\infty(T)$ consists of all bounded functions. The norm on $L^2(T)$ is given by $\|f\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx$. The norm on $L^1(T)$ is given by $\|f\|_1 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|\,dx$. Since the integral is a probability average, their relation is

$$\|f\|_1 \le \|f\|_2 \le \|f\|_\infty. \tag{1.27}$$

Also define the sequence spaces

$$\ell^1 \subset \ell^2 \subset c_0 \subset \ell^\infty. \tag{1.28}$$

The norm on ℓ^1 is $\|c\|_1 = \sum_n |c_n|$. Then norm on ℓ^2 is given by $\|c\|_2^2 = \sum_n |c_n|^2$. The norms on the last two spaces are the same, that is, $\|c\|_\infty$ is the smallest M such that $|c_n| \le M$. The space c_0 consists of all sequences with limit 0 at infinity. The relation between these norms is

$$\|c\|_\infty \le \|c\|_2 \le \|c\|_1. \tag{1.29}$$

We have seen that the Fourier series theorem gives a perfect correspondence between $L^2(T)$ and ℓ^2. For the other spaces the situation is more complex.

Riemann-Lebesgue Lemma. If f is in $L^1(T)$, then the Fourier coefficients of f are in c_0, that is, they approaches 0 at infinity.

Proof: Each function in $L^2(T)$ has Fourier coefficients in ℓ^2, so each function in $L^2(T)$ has Fourier coefficients that vanish at infinity. The map from a function to its Fourier coefficients gives a continuous map from $L^1(T)$ to ℓ^∞. However

every function in $L^1(T)$ may be approximated arbitrarily closely in $L^1(T)$ norm by a function in $L^2(T)$. Hence its coefficients may be approximated arbitrarily well in ℓ^∞ norm by coefficients that vanish at infinity. Therefore the coefficients vanish at infinity.

In summary, the map from a function to its Fourier coefficients gives a continuous map from $L^1(T)$ to c_0. That is, the Fourier coefficients of an integrable function are bounded (this is obvious) and approach zero (Riemann-Lebesgue lemma). Furthermore, the Fourier coefficients determine the function.

The map from Fourier coefficients to functions gives a continuous map from ℓ^1 to $C(T)$. An sequence that is absolutely summable defines a Fourier series that converges absolutely and uniformly to a continuous function.

Theorem. If f is in $L^2(T)$ and if f' exists (in the sense that f is an integral of f) and if f' is also in $L^2(T)$, then the Fourier coefficients are in ℓ^1.

1.4 Pointwise convergence

This gives a very satisfactory picture of Fourier series. However there is one slightly unsatisfying point. The convergence in the L^2 sense does not imply convergence at a particular point. Of course, if the derivative is in L^2 then we have uniform convergence, and in particular convergence at each point. But what if the function is differentiable at one point but has discontinuities at other points? What can we say about convergence at that one point? Fortunately, we can find something about that case by a closer examination of the partial sums.

One looks at the partial sum

$$\sum_{|n|\le N} c_n e^{inx} = \frac{1}{2\pi}\int_0^{2\pi} D_N(x-y) f(y)\, dy. \tag{1.30}$$

Here

$$D_N(x) = \sum_{|n|\le N} e^{inx} = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{1}{2}x)}. \tag{1.31}$$

This Dirichlet kernel $D_N(x)$ has at least some of the properties of an approximate delta function. Unfortunately, it is not positive; instead it oscillates wildly for large N at points away from where $\sin(x/2) = 0$. However the function $1/(2\pi)D_N(x)$ does have integral 1.

Theorem. If for some x the function

$$d_x(z) = \frac{f(x+z) - f(x)}{2\sin(z/2)} \tag{1.32}$$

is in $L^1(T)$, then at that point

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \phi_n(x). \tag{1.33}$$

Note that if $d_x(z)$ is continuous at $z = 0$, then its value at $z = 0$ is $d_x(0) = f'(x)$. So the hypothesis of the theorem is a condition related to differentiability of f at the point x. The conclusion of the theorem is pointwise convergence of the Fourier series at that point. Since f may be discontinuous at other points, it is possible that this Fourier series is not absolutely convergent.

Proof: We have

$$f(x) - \sum_{|n|\le N} c_n e^{inx} = \frac{1}{2\pi} \int_0^{2\pi} D_N(z)(f(x) - f(x-z))\, dz. \tag{1.34}$$

We can write this as

$$f(x) - \sum_{|n|\le N} c_n e^{inx} = \frac{1}{2\pi} \int_0^{2\pi} 2\sin((N+\frac{1}{2})z) d_x(-z)\, dz. \tag{1.35}$$

This goes to zero as $N \to \infty$, by the Riemann-Lebesgue lemma.

1.5 Problems

1. Let $f(x) = x$ defined for $-\pi \le x < \pi$. Find the $L^1(T)$, $L^2(T)$, and $L^\infty(T)$ norms of f, and compare them.

2. Find the Fourier coefficients c_n of f for all n in Z.

3. Find the ℓ^∞, ℓ^2, and ℓ^1 norms of these Fourier coefficients, and compare them.

4. Use the equality of L^2 and ℓ^2 norms to compute

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

5. Compare the ℓ^∞ and L^1 norms for this problem. Compare the L^∞ and ℓ^1 norms for this problem.

6. Use the pointwise convergence at $x = \pi/2$ to evaluate the infinite sum

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1},$$

regarded as a limit of partial sums. Does this sum converge absolutely?

7. Let $F(x) = \frac{1}{2}x^2$ defined for $-\pi \le x < \pi$. Find the Fourier coefficients of this function.

8. Use the equality of L^2 and ℓ^2 norms to compute

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

9. Compare the ℓ^∞ and L^1 norms for this problem. Compare the L^∞ and ℓ^1 norms for this problem.

10. At which points x of T is $F(x)$ continuous? Differentiable? At which points x of T is $f(x)$ continuous? Differentiable? At which x does $F'(x) = f(x)$? Can the Fourier series of $f(x)$ be obtained by differentiating the Fourier series of $F(x)$ pointwise? (This last question can be answered by inspecting the explicit form of the Fourier series for the two problems.)