Math 583A
Fall 2001
Final Exam
(open book)

1. How many branch cuts are needed to define $\sqrt{z(z^2-1)(z^2-4)}$ uniquely on a dense open subset of the complex plane? Sketch a suitable choice of branch cuts.

2. Find the radius of convergence of the Taylor series of $1/\cosh(z)$ about $z = 0$.

3. Let C be the contour that runs from $-a$ to $-\epsilon$, in a clockwise semicircle of radius ϵ in the upper half plane, from ϵ to a, and in a counterclockwise semicircle of radius a in the upper half plane. $\epsilon > 0$ is small and $a > 0$ is large. We know how to evaluate the contour integral

$$\int_C \frac{1}{z^2+4}\,dz = \frac{\pi}{2}$$

by residues. Make the change of variable $z = 1/w$ and express the differential form and the contour in terms of w. Sketch the new contour. Evaluate this new integral by residues.

4. Is

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} e^{inx} = 1 + 2\sum_{n=1}^{\infty} \frac{1}{1+n^2}\cos(nx)$$

a function in $L^1(0, 2\pi)$? Explain.

5. Let $g(x) = \pi - x$ on the interval from 0 to 2π and be 2π periodic. The Fourier series of g is

$$g(x) = \sum_{n\neq 0} \frac{1}{in} e^{inx} = \sum_{n=1}^{\infty} \frac{2}{n}\sin(nx).$$

Let K be the linear transformation on $L^2(0, 2\pi)$ defined by

$$(Kf)(x) = \frac{1}{2\pi}\int_0^{2\pi} g(x-y)f(y)\,dy.$$

Show that the functions $\phi_m(x) = e^{imx}$ are eigenvectors of K, so that

$$(K\phi_m)(x) = \lambda_m \phi_m(x),$$

and find the corresponding eigenvalues λ_m.

6. Consider the matrix

$$U = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}.$$

How many eigenvalues λ does it have in the open unit disk $|\lambda| < 1$?

7. a. Show that for each g in $L^2(R)$ the equation

$$-\frac{d^2u}{dx^2} + u = g(x)$$

has a solution in $L^2(R)$. Find it. Hint: Use Fourier transforms.

b. Let p be a polynomial satisfying the "ellipticity" condition $|p(k)| \geq \epsilon > 0$ for all real k. Show that for each g in $L^2(R)$ the equation

$$p(\frac{1}{i}\frac{d}{dx})u = g(x)$$

has a solution in $L^2(R)$. Hint: In the previous part we had $p(k) = k^2 + 1$.

8. Consider the system of differential equations

$$\frac{du}{dt} = 2u + v^2$$

$$\frac{dv}{dt} = v.$$

The only fixed point is at $u = 0, v = 0$, and the linearization at this fixed point is the same system without the nonlinear v^2 term. It is easy to see that the linearized equation has a very special feature: every solution curve lies on a parabola $u = Cv^2$. Thus for the linearization u is a smooth function of v, that is, all derivatives exist. Show that this is not true for the original non-linear system. Hint: Solve this non-linear system explicitly.