4.2 Optimization

Example Find the global extrema of $f(x) = x^3 - 9x^2 - 48x + 52$ on the following intervals:

(a) $-5 \leq x \leq 12$

(b) $-5 \leq x \leq 14$

(c) $-5 \leq x < \infty$

Example For a positive constant b, the surge function $f(t) = te^{-bt}$ gives the quantity of a drug in the body for time $t \geq 0$.

(a) Find the global extrema of $f(t)$ for $t \geq 0$.

(b) Find the value of b making $t = 10$ the global maximum.

Example When an arrow is shot into the air, its range, R, is defined as the horizontal distance from the archer to the point where the arrow hits the ground. If the ground is horizontal and we neglect air resistance, it can be shown that

$$R = \frac{v_0 \sin(2\theta)}{g},$$

where v_0 is the initial velocity of the arrow, g is the (constant) acceleration due to gravity, and θ is the angle above horizontal, so $0 \leq \theta \leq \pi/2$. What angle θ maximizes R?

Example An object on a spring oscillates about its equilibrium position at $y = 0$. Its distance from equilibrium is given as a function of time, t, by

$$y = e^{-t} \cos t.$$

Find the greatest distance the object goes above and below the equilibrium for $t \geq 0$.

Example The potential energy, U, of a particle moving along the x-axis is given by

$$U = b\left(\frac{a^2}{x^2} - \frac{a}{x}\right),$$

where a and b are positive constants and $x > 0$. What values of x minimizes the potential energy?

Example When an electric current passes through two resistors with resistance r_1 and r_2, connected in parallel, the combined resistance, R, can be calculated from the equation

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2},$$

where R, r_1 and r_2 are positive. Assume that r_2 is constant.

(a) Show that R is an increasing function of r_1.

(b) Where on the interval $a \leq r_1 \leq b$ does R take its maximum value?

Example Two points on the curve $y = \dfrac{x^3}{1+x^4}$ have opposite x values, x and $-x$. Find the points making the slope of the line joining them greatest.

4.3 Optimization and Modeling

Example A. What are the dimensions of an aluminum can that holds 40 in^3 of juice that uses the least material (i.e., aluminum)? Assume that the can is cylindrical, and is capped on both ends.

Example 1. You are running late and want to get to the bus stop as quickly as possible. The bus stop is across a grassy park, 2000 feet west and 600 feet north of your starting position. You can walk west along the edge of the park on the sidewalk at a speed of 6 ft./sec. You can also travel through the grass, but only at a rate of 4 ft./sec. What path should you take to get to the bus stop the fastest?

Example 2. The below figure shows that curves $y = \sqrt{x}, y = 0$ and a rectangle with vertical sides at $x = a$ and $x = 9$. Find the dimensions of the rectangle having the maximum possible area.

Example 3. A closed box has a fixed surface area, A, and a square base with side x.

(a) Find a formula for the volume, V, of the box as a function of x. What is the domain of V?

(b) Find the maximum value of V.

Example 4. A light is suspended at a height h above the floor. The illumination at the point P is inversely proportional to the square of the distance from the point P to the light and directly proportional to the cosine of the angle θ. How far from the floor should the light be to maximize the illumination at the point P?

Example 5. A local club is arranging a charter flight to Hawaii. The cost of the trip is \$1600 each for 90 passengers, with a refund of \$10 per passenger for each passenger in excess of 90. Find the maximum revenue. What is the cost for each passenger to obtain this maximum revenue? What is the number of passengers that will maximize the revenue?

Example 6. In planning a restaurant, it is estimated that a profit of \$8 per seat will be made if the number of seats is no more than 50, inclusive. On the other hand, the profit on each seat will decrease by 10 cents for each seat above 50. What is the maximum profit? How many seats will produce this profit?

Example 7. A window consisting of a rectangle topped by a semicircle is to have an outer perimeter P. Find the radius of the semicircle if the area of the window is to be a maximum.

Example 8. A rectangular field is bounded to the west by a river, and the southwest side is partially bounded by a building that is 20 feet long. What are the dimensions of the field that will have the maximum area with 1000 feet of fencing. (We can assume fencing is not needed along the river nor the building).

Example 9. A company manufactures cylindrical barrels to store nuclear waste. The top and bottom of the barrels are to be made with material that costs \$10 per square foot and the rest is made with material that costs \$8 per square foot. If each barrel is to hold 5 cubic feet, find the dimensions of the barrel that will minimize the total cost.

Example 10. A pigeon is released from a boat (point B) floating on a lake. Because of falling air over the cool water, the energy required to fly one meter over the lake is twice the corresponding energy e required for flying over the bank ($e = 3$ joule/meter). To minimize the energy required to fly from B to the loft, L, the pigeon heads to a point Pon th bank and then flies along the bank to L. The distance $\overline{AL}$ is 2000m, and $\overline{AB}$ is 500m. What is the optimal angle θ? Does this answer change if $\overline{AL}$, $\overline{AB}$ and e have different numerical values?

4.6 Rates and Related Rates

Example A skydiver of mass m jumps from a plane at time $t = 0$. Under certain assumptions, the distance, $s(t)$, he has fallen in time t is given by

$$s(t) = \frac{m^2 g}{k^2}\left(\frac{kt}{m} + e^{-kt/m} - 1\right) \quad \text{for some positive constant } k.$$

(a) Find $s'(0)$ and $s''(0)$ and interpret in terms of the skydiver.

(b) Relate the units of $s'(t)$ and $s''(t)$ to the units of t and $s(t)$.

Example The following graph shows the fuel consumption, g, in miles per gallon (mpg), of a car traveling at v mph. At one moment, the car was going 70 mph and its deceleration was 8000 miles/hour^2. How fast was the fuel consumption changing at that moment? Include units!

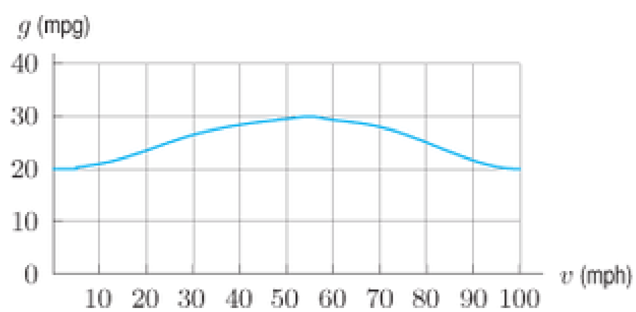

Figure 4.83 Fuel consumption versus velocity

Example

(a) A 3-meter ladder stands against a high wall. The foot of the ladder moves outward at a constant speed of 0.1 meter/sec. When the foot is 1 meter from the wall, how fast is the top of the ladder falling? What about when the foot is 2 meters from the wall?

(b) If the foot of the ladder moves out at a constant speed, how does the speed at which the top falls change as the foot gets farther out?

Example An airplane, flying at 450 km/hr at a constant altitude of 5 km, is approaching a camera mounted on the ground. Let θ be the angle of elevation above the ground at which the camera is pointed. When $\theta = \frac{\pi}{3}$, how fast does the camera have to rotate to keep the plane in view?