4.2 Optimization

Focus on the process in this section, as section 4.3 uses the same methods, but you will need to come up with your own function to optimize.

- f has a **global minimum** at p if $f(p)$ is less than or equal to all values of f.
- f has a **global maximum** at p if $f(p)$ is greater than or equal to all values of f.

Note: Global maxima and minima are sometimes called *extremas* or *optimal values*.

Theorem 4.2: The Extreme Value Theorem
If f is continuous on the closed interval $a \leq x \leq b$, then f has a global maximum and a global minimum on that interval.

Test Candidates for Global Extrema
For a continuous function f on a closed interval $a \leq x \leq b$:

1. Find the critical points of f in the interval.
2. Evaluate the function at the critical points and at the endpoints, a and b.

Warning! If the function is defined on an open interval $a < x < b$ or on all real numbers, there may or may not be a global minimum or maximum. For example:

Global Extrema on an Open Interval or on All Real Numbers
For a continuous function, f, find the value of f at all the critical points and sketch a graph. Look at values of f when x approaches the endpoints of the interval, or approaches $\pm\infty$, as appropriate. If there is only one critical point, look at the sign of f' on either side of the critical point.

Finding Upper and Lower Bounds
A problem which is closely related to finding maxima and minima is finding the **bounds** of a function. It will always be the case that the *best possible bounds* for a function, f, is given by

$$\text{global minimum} \leq f(x) \leq \text{global maximum}.$$

We could always make trivial upper and lower bounds, for instance, $-5 \leq \sin x \leq 2$, but we can do much better, $-1 \leq \sin x \leq 1$.

4.3 Optimization and Modeling

(Most challenging section! Be careful, WebAssign sometimes walks through the steps of these problems... you will need to know how to do the problem without guidance.)

Practical Tips for Modeling Optimization Problems

1. Make sure that you know what quantity or function is to be optimized.
2. If possible, make several sketches showing how the elements that vary are related. Label your sketches clearly by assigning variables to quantities which change.
3. Try to obtain a formula for the function to be optimized in terms of the variables that you identified in the previous step. If necessary, eliminate from this formula all but one variable. Identify the domain over which this variable varies.
4. Find the critical points and evaluate the function at these points and at the endpoints (if relevant) to find the global maxima and minima.

4.4 Families of Functions and Modeling

We say that all functions of the form $y = a(x+b)^2 + c$ form a **family of functions**; their graphs are like that of $y = x^2$, except for shifts and stretches determined by the values of a, b and c. The constants a, b, c are called **parameters**. Different values of the parameters give different members of the family.

One reason for studying families of functions is their use in mathematical modeling. Confronted with the problem of modeling some phenomenon, a crucial first step involves recognizing families of functions which might fit the available data.

Motion Under Gravity

The position of an object moving vertically under the influence of gravity can be described by a function in the two parameter family

$$y = -4.9t^2 + v_0 t + y_0,$$

where t is the time in seconds and y is the distance in meters above the ground. Why are the parameters v_0 and y_0 important here? When will the object reach its maximum height?

The Bell-Shaped Curve

This family is related to the **normal density function**, used in probability and statistics. The family is given by

$$y = e^{-(x-a)^2/b},$$

where we assume that $b > 0$.

To see the role of the parameter a, fix $b = 1$, then the subfamily becomes

$$y = e^{-(x-a)^2}.$$

Thus, the role of a is to shift the graph of $y = e^{-x^2}$ to the right or left.

We now consider the role of the parameter b by studying the family with $a = 0$:

$$y = e^{-x^2/b}.$$

Let us investigate the critical points and the points of inflection of these curves. We calculate

$$\frac{dy}{dx} = -\frac{2x}{b} e^{-x^2/b},$$

and,

$$\frac{d^2y}{dx^2} = \frac{2}{b}\left(\frac{2x^2}{b} - 1\right)e^{-x^2/b}.$$

The only critical point occurs at $x = 0$. At that point, $y = 1$ and by the second derivative test, there is a local maximum at $x = 0$ (this is also a global maximum).

Inflection points when $x = \pm\sqrt{\frac{b}{2}}$. Looking at the expression for d^2y/dx^2, we see that d^2y/dx^2 is negative for $x = 0$, and positive as $x \to \pm\infty$. Therefore, the concavity changes at $x = -\sqrt{b/2}$ and $x = \sqrt{b/2}$, so we have inflection points here.

Returning to the two-parameter family, $y = e^{-(x-a)^2/b}$, we conclude there is a maximum at $x = a$, obtained by horizontally shifting the maximum $x = 0$ of $y = e^{-x^2/b}$ by a units. There are inflection points at $x = a \pm \sqrt{b/2}$, obtained by shifting the inflection points $x = \pm\sqrt{b/2}$ of $y = e^{-x^2/b}$ by a units.

With this information we can see the effect of the parameters. The parameter a determines the location of the center of the bell and the parameter b determines how narrow or wide the bell is. If b is small, then the inflection points are close to a and the bell is sharply peaked near n; if b is large, the inflection points are farther away from a and the bell is spread out.

Problems in this section roughly fall into one of three categories:

1. Investigate the effect of parameters on the graphs,
2. Find values of the parameters under certain conditions,
3. Work with critical points and inflection points in terms of parameters.

4.6 Rates and Related Rates

Derivatives represent rates of change. In this section, we see how to calculate rates in a variety of situations. Some problems are rates of change as you have seen in the past. Other problems are related rates. ('Students often make mistakes because they want to write down everything they know in their picture.')

Example A spherical snowball is melting. Its radius decreases at a constant rate of 2 cm per minute from an initial value of 70 cm. How fast is the volume decreasing half an hour later?

Related Rates

Above the radius of the snowball decreased at a constant rate. A more realistic scenario is for the radius to decrease at different rates. Then, we may not be able to write a formula for V as a function of t. However, we still may be able to calculate $\frac{dV}{dt}$, as in the following example.

Example A spherical snowball melts in such a way that the instant at which its radius is 20 cm, its radius is decreasing at 3 cm/min. What what rate is the volume of the ball changing at that instant?

4.7 L'Hôpital's Rule, Growth, and Dominance

(Just because you are given a rule does NOT mean you can apply it with everything. The more important issue for you is to know when the rule does not apply.)

Suppose we want to calculate the exact value of the limit

$$\lim_{x\to 0}\frac{e^{2x}-1}{x}.$$

Clearly we cannot substitute $x = 0$, because this would give us $0/0$, which is undefined. In the past we have instead substituted values of x near 0 to obtain an approximation to this limit.

However, the limit can be calculated exactly using local linearity! Suppose we let $f(x)$ be the numerator and $g(x)$ be the denominator, i.e.,

$$f(x) = e^{2x} - 1 \quad g(x) = x.$$

Then, $f(0) = 0$ and $f'(x) = 2x^{2x}$, so $f'(0) = 2$. Hence, the tangent line to $f(x)$ at $x = 0$ is given by $y = 2x$. Since $g(x) = x$ is already linear, there is no need to approximate it, but notice that $g'(x) = 1$. We are now interested in the ratio of $f(x)/g(x)$, which is approximately the ratios of the y-values in our local linearization. Hence,

$$\frac{f(x)}{g(x)} = \frac{e^{2x}-1}{x} \approx \frac{2x}{x} = \frac{2}{1} = \frac{f'(0)}{g'(0)}.$$

Since the approximation given by local linearization gets better as the x-values approach the point, it follows that as $x \to 0$ the approximation actually turns into an equality! Thus we have

$$\lim_{x\to 0}\frac{e^{2x}-1}{x} = \frac{2}{1} = 2.$$

L'Hôpital's Rule

If f and g are differentiable, $f(a) = g(a) = 0$, and $g'(a) \neq 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

More generally, if f and g are differentiable and $f(a) = g(a) = 0$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided the limit on the right exists.

We can also use L'Hôpital's Rule in cases involving infinity!

L'Hôpital's Rule on Limits with Infinity

L'Hôpital's rule applies to limits involving infinity provided f and g are differentiable. For a any real number or $\pm\infty$:

- When $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$,

or

- When $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$,

it can be shown that:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided the limit on the right-hand side exists.

Dominance: Powers, Polynomials, Exponentials, and Logarithms

In Chapter 1 we saw that some functions grew much faster than other as $x \to \infty$. We say that g **dominates** f as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. L'Hôpital's Rule gives us an easy way of checking this.

For instance, check that $x^{1/2}$ dominates $\ln x$ as $x \to \infty$.

Recognizing the Form of a Limit

Although expressions like $0/0$ and ∞/∞ have no numerical value, they are useful in describing the form of a limit. We can use L'Hôpital's Rule to calculate some limits of the form $\lim_{x\to\infty} f(x)g(x) = 0 \cdot \infty$, providing we rewrite them appropriately.

Note, we can use L'Hôpital's Rule after rewriting the forms:

$$\frac{0}{0}, \frac{\infty}{\infty}, \infty \cdot 0, \infty - \infty, 1^{\infty}, 0^{0}, \infty^{0}$$