Section 7.1
Recognizing Integrals

For each integral, decide which of the following is needed: (1) substition; (2) algebra or a trig. identity; (3) nothing needed; or (4) can't be done using techniques from Calculus I. Then, evaluate each integral (except for type (4), of course!).

1. - $\int \left(x^3+1\right) dx$
   - $\int x^2 \left(x^3+1\right)^4 dx$
   - $\int \sqrt{x^3+1} dx$
   - $\int \left(x^3+1\right)^2 dx$

2. - $\int \sqrt{x} \left(1-x^2\right) dx$
   - $\int \sqrt{1-x^2} dx$
   - $\int \frac{1}{\sqrt{1-x^2}} dx$
   - $\int \frac{x}{\sqrt{1-x^2}} dx$

3. - $\int \cos^2 x \sin^3 x dx$
   - $\int \sqrt{1-\cos^2 x} dx$
   - $\int \frac{dx}{\cos^2 x}$
   - $\int \frac{dx}{\cos x \sqrt{\sin x}}$

4. - $\int \tan x \sec x dx$
   - $\int \tan x \cos x dx$
   - $\int \frac{\sec^2 x}{\sqrt{\tan x}} dx$
   - $\int \frac{dx}{\tan x + 1}$

5. - $\int e^{-x^2} dx$
   - $\int \frac{e^x}{3+e^x} dx$
   - $\int \left(e^x+3\right) dx$
   - $\int \frac{\ln\left(e^{2x}\right)}{x^2} dx$

Solutions.

For each integral, decide which of the following is needed: (1) substition; (2) algebra or a trig. identity; (3) nothing needed; or (4) can't be done using techniques from Calculus I. Then, evaluate each integral (except for type (4), of course!).

1. - $\int \left(x^3+1\right) dx$

     Type (3).

     $$\int \left(x^3+1\right) dx = \frac{1}{4}x^4 + x + C$$

   - $\int x^2 \left(x^3+1\right)^4 dx$

     Type (1). Let $u = x^3+1$ so that $du = 3x^2 dx$.

     $$\begin{aligned}\int x^2 \left(x^3+1\right)^4 dx &= \frac{1}{3}\int u^4 du \\ &= \frac{1}{3}\cdot\frac{1}{5}u^5 + C \\ &= \frac{1}{15}\left(x^3+1\right)^5 + C.\end{aligned}$$

   - $\int \sqrt{x^3+1} dx$

     Type (4).

   - $\int \left(x^3+1\right)^2 dx$

     Type (2).

     $$\begin{aligned}\int \left(x^3+1\right)^2 dx &= \int \left(x^6+2x^3+1\right) dx \\ &= \frac{1}{7}x^7 + \frac{1}{2}x^4 + x + C\end{aligned}$$

2. - $\int \sqrt{x}\left(1-x^2\right) dx$

     Type (2).

     $$\begin{aligned}\int \sqrt{x}\left(1-x^2\right) dx &= \int \left(x^{1/2} - x^{5/2}\right) dx \\ &= \frac{2}{3}x^{3/2} - \frac{2}{7}x^{7/2} + C.\end{aligned}$$

   - $\int \sqrt{1-x^2} dx$

     Type (4).

- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}dx$

  Type (3).

$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$$

- $\displaystyle\int \frac{x}{\sqrt{1-x^2}}dx$

  Type (1). Let $u = 1 - x^2$ so that $du = -2xdx$.

$$\begin{aligned} \int \frac{x}{\sqrt{1-x^2}}dx &= \int \frac{1}{\sqrt{u}} \cdot -\frac{1}{2}du \\ &= -\frac{1}{2}\int u^{-1/2}du \\ &= -\frac{1}{2} \cdot 2u^{1/2} + C \\ &= -\sqrt{1-x^2} + C. \end{aligned}$$

3. - $\displaystyle\int \cos^2 x \sin^3 x dx$

  Type (1) - (after a trig. identity). Let $u = \cos x$ so that $du = -\sin x dx$.

$$\begin{aligned} \int \cos^2 x \sin^3 x dx &= \int \cos^2(x)\left(1-\cos^2 x\right)\sin x dx \\ &= -\int u^2\left(1-u^2\right)du \\ &= -\int u^2 - u^4 du \\ &= -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C \\ &= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C. \end{aligned}$$

- $\displaystyle\int \sqrt{1-\cos^2 x}dx$

  Type (2).

$$\begin{aligned} \int \sqrt{1-\cos^2 x}dx &= \int \sqrt{\sin^2 x}dx \\ &= \int \sin x dx \\ &= -\cos x + C. \end{aligned}$$

- $\displaystyle\int \frac{dx}{\cos^2 x}$

  Type (3).

$$\int \frac{dx}{\cos^2 x} = \int \sec^2 x dx = \tan x + C$$

- $\displaystyle\int \frac{dx}{\cos x\sqrt{\sin x}}$

  Type (4).

4. - $\displaystyle\int \tan x \sec x dx$

   Type (3).

   $$\int \tan x \sec x dx = \sec x + C$$

   - $\displaystyle\int \tan x \cos x dx$

   Type (2).

   $$\begin{aligned}\int \tan x \cos x dx &= \int \frac{\sin x}{\cos x}\cos x dx\\ &= \int \sin x dx\\ &= -\cos x + C.\end{aligned}$$

   - $\displaystyle\int \frac{\sec^2 x}{\sqrt{\tan x}}dx$

   Type (1). Let $u = \tan x$ so that $du = \sec^2 x dx$.

   $$\begin{aligned}\int \frac{\sec^2 x}{\sqrt{\tan x}}dx &= \int \frac{1}{\sqrt{u}}du\\ &= 2\sqrt{u} + C\\ &= 2\sqrt{\tan x} + C.\end{aligned}$$

   - $\displaystyle\int \frac{dx}{\tan x + 1}$

   Type (4).

5. - $\displaystyle\int e^{-x^2} dx$

   Type (4).

   - $\displaystyle\int \frac{e^x}{3+e^x}dx$

   Type (1). Let $u = 3 + e^x$ so that $du = e^x dx$.

   $$\begin{aligned}\int \frac{e^x}{3+e^x}dx &= \int \frac{1}{u}du\\ &= \ln|u| + C\\ &= \ln|3+e^x| + C.\end{aligned}$$

- $\int (e^x + 3)\, dx$

  Type (3).

  $$\int (e^x + 3)\, dx = e^x + 3x + C$$

- $\int \frac{\ln\left(e^{2x}\right)}{x^2} dx$

  Type (2).

  $$\begin{aligned} \int \frac{\ln\left(e^{2x}\right)}{x^2} dx &= \int \frac{2x}{x^2} dx \\ &= \frac{2}{x} dx \\ &= 2\ln|x| + C. \end{aligned}$$