Section 7.1

Examples

1. Evaluate each of the following pairs of integrals. Try "chain rule backwards" when possible.

   (a) $\int \frac{1}{x} dx$ and $\int \frac{1}{2x+1} dx$

   (b) $\int \cos\theta d\theta$ and $\int \cos\left(\frac{\pi}{3}\theta\right) d\theta$

   (c) $\int \sqrt{y} dy$ and $\int \sqrt{5-y} dy$

   (d) $\int e^t dt$ and $\int e^{3t} dt$

2. Which of the following could use a substitution? Evaluate each. Show all steps.

   (a) $\int (3y+4)^8 dy$

   (b) $\int (z^3-2)^2 dz$

   (c) $\int 5x \cdot 3^{x^2} dx$

   (d) $\int \tan(3\theta) d\theta$

   (e) $\int \frac{1-x^2}{x^2} dx$

   (f) $\int \frac{p}{1+p^2} dp$

   (g) $\int \frac{dt}{16t^2+1}$

   (h) $\int \frac{\cos(A\alpha)}{(1-\sin(A\alpha))^3} d\alpha$

   (i) $\int \frac{\sec^2\left(\frac{3}{x}\right)}{x^2} dx$

   (j) $\int y^5 \sqrt{1+y^2} dy$

3. Suppose we know that $\int_3^7 g(t)dt = -6$. Evaluate each of the following:

   (a) $\int_{-3}^{-1} \frac{g(2t+9)}{5} dt$

   (b) $\int_2^{\sqrt{8}} xg(x^2-1) dx$

Solutions.

1. Evaluate each of the following pairs of integrals. Try "chain rule backwards" when possible.

   (a) $\int \frac{1}{x} dx$ and $\int \frac{1}{2x+1} dx$

   $$\int \frac{1}{x} dx = \ln|x| + C \qquad \text{and} \qquad \int \frac{1}{2x+1} dx = \frac{1}{2} \ln|2x+1| + C$$

   (b) $\int \cos\theta d\theta$ and $\int \cos\left(\frac{\pi}{3}\theta\right) d\theta$

   $$\int \cos\theta d\theta = \sin\theta + C \qquad \text{and} \qquad \int \cos\left(\frac{\pi}{3}\theta\right) d\theta = \frac{3}{\pi} \sin\left(\frac{\pi}{3}\theta\right) + C$$

   (c) $\int \sqrt{y} dy$ and $\int \sqrt{5-y} dy$

   $$\int \sqrt{y} dy = \frac{2}{3} y^{3/2} + C \qquad \text{and} \qquad \int \sqrt{5-y} dy = -\frac{2}{3}(5-y)^{3/2} + C$$

   (d) $\int e^t dt$ and $\int e^{3t} dt$

   $$\int e^t dt = e^t + C \qquad \text{and} \qquad \int e^{3t} dt = \frac{1}{3} e^{3t} + C$$

2. Which of the following could use a substitution? Evaluate each. Show all steps.

   (a) $\int (3y+4)^8 dy$

   $$\int (3y+4)^8 dy = \frac{1}{3} \cdot \frac{1}{9}(3y+4)^9 + C = \frac{1}{27}(3y+4)^9 + C \qquad \text{(chain rule backwards)}$$

   (b) $\int (z^3-2)^2 dz$

   $$\int (z^3-2)^2 dz = \int z^6 - 4z^3 + 4 dz = \frac{1}{7} z^7 - z^4 + 4z + C \qquad \text{(algebra)}$$

   (c) $\int 5x \cdot 3^{x^2} dx$

   Substitution: $u = x^2$ so that $du = 2x dx$.

   $$\begin{aligned} \int 5x \cdot 3^{x^2} dx &= 5 \int x \cdot 3^{x^2} dx \\ &= 5 \int e^u \cdot \frac{1}{2} du \\ &= \frac{5}{2} \int 3^u du \\ &= \frac{5}{2} \cdot \frac{1}{\ln 3} 3^u + C \\ &= \frac{5}{2 \ln 3} 3^{x^2} + C. \end{aligned}$$

(d) $\int \tan(3\theta)d\theta$

First we note,

$$\int \tan(3\theta)d\theta = \int \frac{\sin(3\theta)}{\cos(3\theta)}d\theta.$$

Substitution: $u = \cos(3\theta)$ so that $du = -3\sin(3\theta)d\theta$.

$$\begin{aligned}\int \tan(3\theta)d\theta = \int \frac{\sin(3\theta)}{\cos(3\theta)}d\theta &= \int \frac{1}{u} \cdot -\frac{1}{3}du \\ &= -\frac{1}{3}\int \frac{1}{u}du \\ &= -\frac{1}{3}\ln|u| + C \\ &= -\frac{1}{3}\ln|\cos(3\theta)| + C.\end{aligned}$$

(e) $\int \frac{1-x^2}{x^2}dx$

$$\int \frac{1-x^2}{x^2}dx = \int \left(\frac{1}{x^2} - 1\right)dx = -\frac{1}{x} - x + C \qquad \text{(algebra)}$$

(f) $\int \frac{p}{1+p^2}dp$

Substitution: $u = 1 + p^2$ so that $du = 2pdp$.

$$\begin{aligned}\int \frac{p}{1+p^2}dp &= \int \frac{1}{u} \cdot \frac{1}{2}du \\ &= \frac{1}{2}\int \frac{1}{u}du \\ &= \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln|1+p^2| + C.\end{aligned}$$

(g) $\int \frac{dt}{16t^2+1}$

First, we note

$$\int \frac{dt}{16t^2+1} = \int \frac{dt}{(4t^2)+1}dt.$$

Substitution: $u = 4t$ so that $du = 4dt$.

$$\begin{aligned}\int \frac{dt}{16t^2+1} = \int \frac{dt}{(4t^2)+1}dt &= \frac{1}{4}\int \frac{1}{u^2+1}du \\ &= \frac{1}{4}\arctan(u) + C \\ &= \frac{1}{4}\arctan(4t) + C.\end{aligned}$$

(h) $\displaystyle\int \frac{\cos(A\alpha)}{(1-\sin(A\alpha))^3}d\alpha$

Substitution: $u = 1-\sin(A\alpha)$ so that $du = -A\cos(A\alpha)d\alpha$.

$$\begin{aligned}
\int \frac{\cos(A\alpha)}{(1-\sin(A\alpha))^3}d\alpha &= -\frac{1}{A}\int \frac{1}{u^3}du \\
&= -\frac{1}{A}\cdot -\frac{1}{2u^2} + C \\
&= \frac{1}{2A\sin^2(A\alpha)} + C.
\end{aligned}$$

(i) $\displaystyle\int \frac{\sec^2\left(\frac{3}{x}\right)}{x^2}dx$

First, we note

$$\int \frac{\sec^2\left(\frac{3}{x}\right)}{x^2}dx = \int \sec^2\left(3x^{-1}\right)\cdot x^{-2}dx.$$

Substitution: $u = 3x^{-1}$ so that $du = -3x^{-2}dx$.

$$\begin{aligned}
\int \frac{\sec^2\left(\frac{3}{x}\right)}{x^2}dx = \int \sec^2\left(3x^{-1}\right)\cdot x^{-2}dx &= -\frac{1}{3}\int \sec^2 u du \\
&= -\frac{1}{3}\tan u + C \\
&= -\frac{1}{3}\tan\left(3x^{-1}\right) + C.
\end{aligned}$$

(j) $\displaystyle\int y^5\sqrt{1+y^2}dy$

First, we note

$$\int y^5\sqrt{1+y^2}dy = \int \left(1+y^2\right)^{1/2}(y^2)^2\cdot ydy.$$

Substitution: $u = 1+y^2$ so that $du = 2ydy$ and $y^2 = u-1$.

$$\begin{aligned}
\int y^5\sqrt{1+y^2}dy = \int \left(1+y^2\right)^{1/2}(y^2)^2\cdot ydy &= \frac{1}{2}\int u^{1/2}(u-1)^2du \\
&= \frac{1}{2}\int u^{5/2} - 2u^{3/2} + u^{1/2}du \\
&= \frac{1}{2}\left(\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C \\
&= \frac{1}{2}\left(\frac{2}{7}(1+y^2)^{7/2} - \frac{4}{5}(1+y^2)^{5/2} + \frac{2}{3}(1+y^2)^{3/2}\right) + C.
\end{aligned}$$

3. Suppose we know that $\int_3^7 g(t)dt = -6$. Evaluate each of the following:

(a) $\int_{-3}^{-1} \frac{g(2t+9)}{5} dt$

Substitution: $u = 2t + 9$ so that $du = 2dt$ and the bounds change to $-3 \mapsto 3$ and $-1 \mapsto 7$.

$$\begin{aligned} \int_{-3}^{-1} \frac{g(2t+9)}{5} dt &= \frac{1}{5} \int_{-3}^{-1} g(2t+9)dt \\ &= \frac{1}{5} \cdot \frac{1}{2} \int_3^7 g(u)du \\ &= \frac{1}{10} \cdot (-6) \\ &= -\frac{3}{5}. \end{aligned}$$

(b) $\int_2^{\sqrt{8}} xg(x^2 - 1)dx$

Substitution: $u = x^2 - 1$ so that $du = 2xdx$ and the bounds change to $2 \mapsto 3$ and $\sqrt{8} \mapsto 7$.

$$\begin{aligned} \int_2^{\sqrt{8}} xg(x^2 - 1)dx &= \frac{1}{2} \int_3^7 g(u)du \\ &= \frac{1}{2} \cdot (-6) \\ &= -3. \end{aligned}$$