Section 6.4

Examples

1. What does each of the following represent? Which are numbers and which are functions?

   (a) $\int_a^b f(t)dt$

   (b) $\int f(t)dt$

   (c) $\int_a^x f(t)dt$

   (d) $\frac{d}{dx}\int_a^x f(t)dt$

2. A well-known function, called $\text{erf}(x)$, is defined to be

$$\text{erf}(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-t^2} dt.$$

   (a) Find the exact value of $\text{erf}(0)$.

   (b) Estimate $\text{erf}(3)$ using a left-hand sum with $n = 25$.

   (c) Find $\frac{d}{dx}(\text{erf}(x))$.

   (d) Find $\frac{d}{dx}\left(\text{erf}(x^3)\right)$

   (e) Determine where $\text{erf}(x)$ is increasing and decreasing.

   (f) Determine where $\text{erf}(x)$ is concave up and down.

   (g) Sketch a graph of $\text{erf}(x)$.

Solutions.

1. What does each of the following represent? Which are numbers and which are functions?

   (a) $\int_a^b f(t)dt$

   This is a number. It is the area under the graph of $f(t)$, above the t-axis and between the points $t = a$ and $t = b$.

   (b) $\int f(t)dt$

   This is a family of functions. The derivative of any member of this family equal $f(t)$.

   (c) $\int_a^x f(t)dt$

   This is a function (in the variable x). It represents the area under the graph of $f(t)$, above the t-axis and between the points $t = a$ and $t = x$.

   (d) $\frac{d}{dx}\int_a^x f(t)dt$

   This is a function. It is the function $f(x)$.

2. A well-known function, called $\text{erf}(x)$, is defined to be

$$\text{erf}(x) = \int_0^x \frac{2}{\sqrt{\pi}}e^{-t^2}dt.$$

   (a) Find the exact value of $\text{erf}(0)$.

$$\text{erf}(0) = \int_0^0 \frac{2}{\sqrt{\pi}}e^{-t^2}dt = 0.$$

   (b) Estimate $\text{erf}(3)$ using a left-hand sum with $n = 25$.

$$\text{erf}(3) = \int_0^3 \frac{2}{\sqrt{\pi}}e^{-t^2}dt \approx 1.0677 \text{ using a numerical integration utility.}$$

   (c) Find $\frac{d}{dx}(\text{erf}(x))$.

$$\frac{d}{dx}(\text{erf}(x)) = \frac{d}{dx}\left(\int_0^x \frac{2}{\sqrt{\pi}}e^{-t^2}dt\right) = \frac{2}{\sqrt{\pi}}e{-}x^2 \text{ using the 2nd FTC.}$$

   (d) Find $\frac{d}{dx}\left(\text{erf}(x^3)\right)$

$$\frac{d}{dx}\left(\text{erf}(x^3)\right) = \frac{2}{\sqrt{\pi}}e^{-(x^3)^2}\cdot 3x^2 = \frac{6x^2}{\sqrt{\pi}}e^{-(x^6)} \text{ using the chain rule.}$$

   (e) Determine where $\text{erf}(x)$ is increasing and decreasing.

   In part (c) we found the derivative of $\text{erf}(x)$ to be $\frac{2}{\sqrt{\pi}}e^{-x^2}$, which is always positive. Thus, $\text{erf}(x)$ is always increasing!

(f) Determine where $\mathrm{erf}(x)$ is concave up and down.

Computing the second derivative we find

$$\frac{d^2}{dx^2}\left(\mathrm{erf}(x)\right) = \frac{-4x}{\sqrt{\pi}} e^{-x^2},$$

which is positive for $x < 0$ and negative for $x > 0$. Thus, $\mathrm{erf}(x)$ is concave up and then concave down with $x = 0$ as an inflection point.

(g) Sketch a graph of $\mathrm{erf}(x)$.