Section 6.2

Examples

1. Evaluate each of the following indefinite integrals:

   (a) $\displaystyle\int \frac{1}{\sqrt[3]{t}}dt$

   (b) $\displaystyle\int \pi^5 dy$

   (c) $\displaystyle\int \frac{x^2+3}{x}dx$

   (d) $\displaystyle\int \left(5^z - \frac{1}{z^2+1} + \sec z \tan z\right) dz$

2. Find $F(x)$ if $f(x) = \sin x$ and $F(\pi) = 3$.

3. Consider the closed region bounded between $f(x) = 8(3-x)\ln x$ and the x-axis.

   (a) Verify that $F(x) = 2x^2 - 24x - 4x^2 \ln x + 24x \ln x$ is an antiderivative of $f(x)$.

   (b) Find the exact area bounded between $f(x)$ and the x-axis. Include a sketch.

4. Find the exact area (include a sketch)

   (a) under $f(t) = \dfrac{1}{\sqrt{1-t^2}}$ over $\left[0, \dfrac{1}{2}\right]$

   (b) between $y = \dfrac{3}{4}x + 3$ and $y = 16 - x^2$.

5. Find the average value of the function $g(x) = x^3(5-x)$ between its x-intercepts.

6. Find the value of b so that $\displaystyle\int_1^b \frac{6}{x^2}dx = 3$.

7. Water is entering a large tank at the rate of $r(t) = 3\sqrt{t} + 2$ liters/hr. Find the exact amount of water that entered the tank during the first day.

Solutions.

1. Evaluate each of the following indefinite integrals:

   (a) $\displaystyle\int \frac{1}{\sqrt[3]{t}}dt$

   $$\int \frac{1}{\sqrt[3]{t}}dt = \int t^{-1/3}dt = \frac{3}{2}t^{2/3} + C$$

   (b) $\displaystyle\int \pi^5 dy$

   $$\int \pi^5 dy = \pi^5 y + C$$

   (c) $\displaystyle\int \frac{x^2+3}{x}dx$

   $$\int \frac{x^2+3}{x}dx = \int \left(x + \frac{3}{x}\right)dx = \frac{1}{2}x^2 + 3\ln|x| + C$$

   (d) $\displaystyle\int \left(5^z - \frac{1}{z^2+1} + \sec z \tan z\right)dz$

   $$\int \left(5^z - \frac{1}{z^2+1} + \sec z \tan z\right)dz = \frac{1}{\ln 5}5^z - \arctan z + \sec z + C$$

2. Find $F(x)$ if $f(x) = \sin x$ and $F(\pi) = 3$.

   We know $F(x) = -\cos x + C$, so we simply need to determine the correct value of C. Since

   $$\begin{aligned} 3 &= F(\pi) \\ &= -\cos(\pi) + C \\ &= 1 + C, \end{aligned}$$

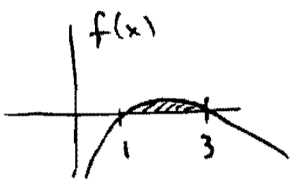
   it follows $C = 2$. Thus, $F(x) = -\cos(x) + 2$.

3. Consider the closed region bounded between $f(x) = 8(3-x)\ln x$ and the x-axis.

   (a) Verify that $F(x) = 2x^2 - 24x - 4x^2\ln x + 24x\ln x$ is an antiderivative of $f(x)$.

   $$\begin{aligned} F'(x) &= 4x - 24 - \left(4x^2 \cdot \frac{1}{x} + \ln x \cdot 8x\right) + \left(24x \cdot \frac{1}{x} + \ln x \cdot 24\right) \\ &= 4x - 24 - 4x - 8x\ln x + 24 + 24\ln x \\ &= 8\ln x(3-x) = f(x). \end{aligned}$$

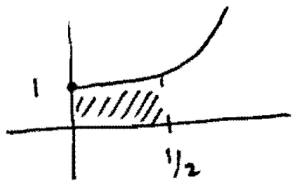
   (b) Find the exact area bounded between $f(x)$ and the x-axis. Include a sketch.

$$\begin{aligned}\int_1^3 8(3-x)\ln x\,dx &= F(3)-F(1)\\ &= (-54-36\ln 3+72\ln 3)-(-22)\\ &= -32+36\ln 3.\end{aligned}$$

4. Find the exact area (include a sketch)

(a) under $f(t)=\dfrac{1}{\sqrt{1-t^2}}$ over $\left[0,\dfrac{1}{2}\right]$



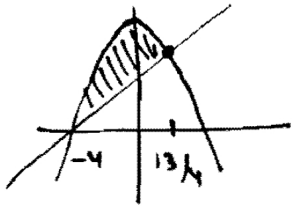
$$\begin{aligned}\int_0^{1/2}\frac{1}{\sqrt{1-t^2}}dt &= \arcsin t\big|_0^{1/2}\\ &= \arcsin\left(\frac{1}{2}\right)-\arcsin(0)\\ &= \frac{\pi}{6}-0\\ &= \frac{\pi}{6}.\end{aligned}$$

(b) between $y=\dfrac{3}{4}x+3$ and $y=16-x^2$.

First, we find the points of intersection by solving

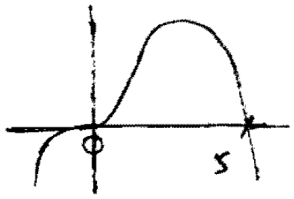
$$\begin{aligned}\frac{3}{4}x+3 &= 16-x^2\\ x^2-\frac{3}{4}x-13 &= 0,\end{aligned}$$

so $x=-4$ and $x=\frac{13}{4}$.



$$\begin{aligned}\int_{-4}^{13/4}\left((16-x^2)-\left(\frac{3}{4}x+3\right)\right)dx &= \int_{-4}^{13/4}\left(-x^2-\frac{3}{4}x+13\right)dx\\ &= \left(-\frac{1}{3}x^3-\frac{3}{8}x^2+13x\right)\Big|_{-4}^{13/4}\\ &= \frac{10309}{384}-\left(-\frac{110}{3}\right)\\ &= \frac{24389}{384}.\end{aligned}$$

5. Find the average value of the function $g(x) = x^3(5-x)$ between its x-intercepts.



$$\begin{aligned}
\text{Average Value} &= \frac{1}{5-0}\int_0^5 x^3(5-x)dx \\
&= \frac{1}{5}\int_0^5 5x^3 - x^4 dx \\
&= \frac{1}{5}\left(\frac{5}{4}x^4 - \frac{1}{5}x^5\right)\Big|_0^5 \\
&= \frac{1}{5}\left(\frac{625}{4}\right) \\
&= \frac{125}{4}.
\end{aligned}$$

6. Find the value of b so that $\displaystyle\int_1^b \frac{6}{x^2}dx = 3$.

$$\begin{aligned}
3 &= \int_1^b \frac{6}{x^2}dx \\
&= \int_1^b 6x^{-2}dx \\
&= -\frac{6}{x}\Big|_1^b \\
&= -\frac{6}{b} + 6
\end{aligned}$$

Hence, $-\dfrac{6}{b} + 6 = 3$, which gives us that $b = 2$.

7. Water is entering a large tank at the rate of $r(t) = 3\sqrt{t} + 2$ liters/hr. Find the exact amount of water that entered the tank during the first day.

First, note that $r(t)$ is being measured in liters/hr, which means it's integral (with respect to time) will be measured in liters!

$$\begin{aligned}
\int_0^{24} 3\sqrt{t} + 2dt &= \int_0^{24} 3t^{1/2} + 2dt \\
&= \left(2t^{3/2} + 2t\right)\Big|_0^{24} \\
&= 2\left(24\right)^{3/2} + 2(24) \\
&= 96\sqrt{6} + 48 \text{ liters.}
\end{aligned}$$