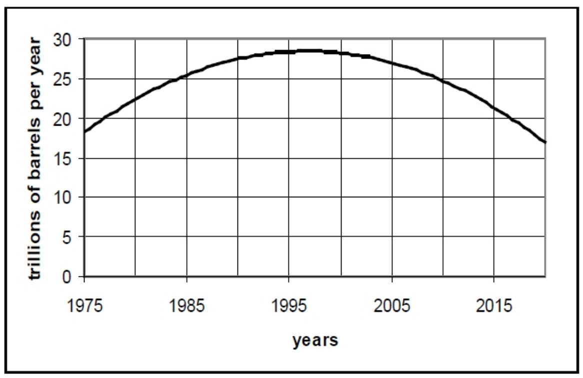
Section 5.4

Average Value Examples

1. Find the average function value of $g(t) = \sqrt{16 - t^2}$ over $[-4, 4]$. Give an exact answer. Hint: Use geometry.

2. Suppose $f(x) = \ln x$ and $F(x) = x \ln x - x$. Confirm the conditions in the Fundamental Theorem of Calculus for these functions on the interval $[1, 3]$. Find the exact average value of $f(x)$.

3. The projected rate of oil production in a particular country up to the year 2020 is shown below. If this projection is correct, estimate the amount of oil that would be produced between 2000 and 2015. Determine the average rate of production during that time interval.



Solutions.

1. Find the average function value of $g(t) = \sqrt{16 - t^2}$ over $[-4, 4]$. Give an exact answer. Hint: Use geometry.

$$\text{Average function value over } [-4,4] = \frac{1}{4-(-4)} \int_{-4}^{4} \sqrt{16-t^2}dt$$

The integral represents the area of the top half of a circle of radius 4. Thus, by geometry we know

$$\text{Average function value over } [-4,4] = \frac{1}{4-(-4)} \int_{-4}^{4} \sqrt{16-t^2}dt = \frac{1}{8} \cdot \frac{1}{2}\pi(4)^2 = \pi.$$

2. Suppose $f(x) = \ln x$ and $F(x) = x \ln x - x$. Confirm the conditions in the Fundamental Theorem of Calculus for these functions on the interval $[1, 3]$. Find the exact average value of $f(x)$.

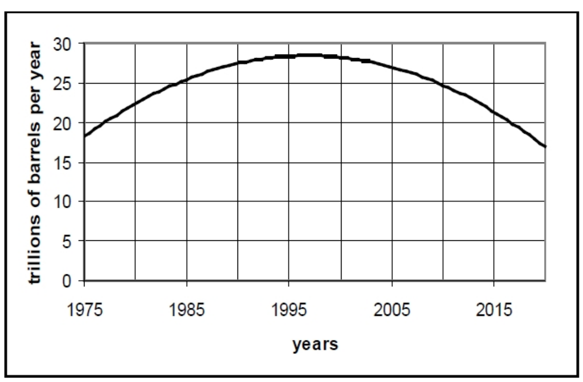
The two conditions of the FTC are satisfied since:

- $f(x) = \ln x$ is continuous on $[1, 3]$,
- $F'(x) = \frac{d}{dx}(x \ln x - x) = x\left(\frac{1}{x}\right) + \ln x - 1 = \ln x = f(x)$.

Thus, using the FTC, we have:

$$\begin{aligned} \text{Average function value over } [1,3] = \frac{1}{3-1}\int_{1}^{3} \ln x dx &= \frac{1}{2}\left(F(3) - F(1)\right) \\ &= \frac{1}{2}\left((3\ln 3 - 3) - (\ln 1 - 1)\right) \\ &= \frac{1}{2}\left(3 \ln 3 - 2\right). \end{aligned}$$

3. The projected rate of oil production in a particular country up to the year 2020 is shown below. If this projection is correct, estimate the amount of oil that would be produced between 2000 and 2015. Determine the average rate of production during that time interval.



Rewriting the average function value formula, we see $\int_{a}^{b} f(x)dx = (\text{average value})\,(b - a)$.

The average value is the height of a single rectangle whose area is the same as the area under the curve over the given interval.

The height of that single rectangle is approximately 25, meaning the average rate of production is approximately 25 trillion barrels per year.