

Section 4.7

More Examples

Evaluate the limits in Problems 1 through 8. If L'Hôpital's Rule applies, you must use it. Express your answers in exact form.

1. $\lim_{\alpha \rightarrow 0} \alpha \cot(2\alpha)$

2. $\lim_{y \rightarrow \infty} y \ln \left(\frac{y+1}{y-1} \right)$

3. $\lim_{x \rightarrow 0^-} x^3 e^{1/x}$

4. $\lim_{x \rightarrow 0} \frac{x^2 + 3x}{\sinh x}$

5. $\lim_{x \rightarrow 1^+} \left(\frac{1}{\ln(1+x)} - \frac{1}{x} \right)$

6. $\lim_{x \rightarrow \infty} (\ln |2x - 4| - \ln |x + 3|)$

7. $\lim_{\theta \rightarrow \infty} \theta \sin \left(\frac{1}{\theta} \right)$

8. $\lim_{y \rightarrow 0} \frac{2^y}{y^2}$

9. Use $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = e$ and substitution to find

(a) $\lim_{t \rightarrow 0^+} (1+t)^{1/t}$

(b) $\lim_{y \rightarrow \infty} \left(1 + \frac{5}{y} \right)^y$

Solutions.

Evaluate the limits in Problems 1 through 8. If L'Hôpital's Rule applies, you must use it. Express your answers in exact form.

1. $\lim_{\alpha \rightarrow 0} \alpha \cot(2\alpha)$

$$\lim_{\alpha \rightarrow 0} \alpha \cot(2\alpha) = 0 \cdot \infty \text{ (Indeterminate form - we need to rewrite!)}$$

$$\lim_{\alpha \rightarrow 0} \alpha \cot(2\alpha) = \lim_{\alpha \rightarrow 0} \frac{\alpha}{\frac{1}{\cot(2\alpha)}} = \lim_{\alpha \rightarrow 0} \frac{\alpha}{\tan(2\alpha)} = \frac{0}{0} \text{ (L'Hôpital's Rule applies!)}$$

$$\lim_{\alpha \rightarrow 0} \alpha \cot(2\alpha) = \lim_{\alpha \rightarrow 0} \frac{\alpha}{\frac{1}{\cot(2\alpha)}} = \lim_{\alpha \rightarrow 0} \frac{\alpha}{\tan(2\alpha)} = \lim_{\alpha \rightarrow 0} \frac{1}{2 \sec^2(2\alpha)} = \frac{1}{2}.$$

2. $\lim_{y \rightarrow \infty} y \ln \left(\frac{y+1}{y-1} \right)$

$$\lim_{y \rightarrow \infty} y \ln \left(\frac{y+1}{y-1} \right) = \infty \cdot 0 \text{ (Indeterminate form - we need to rewrite!)}$$

$$\lim_{y \rightarrow \infty} y \ln \left(\frac{y+1}{y-1} \right) = \lim_{y \rightarrow \infty} \frac{\ln(y+1) - \ln(y-1)}{\frac{1}{y}} = \frac{0}{0} \text{ (L'Hôpital's Rule applies!)}$$

$$\lim_{y \rightarrow \infty} y \ln \left(\frac{y+1}{y-1} \right) = \lim_{y \rightarrow \infty} \frac{\ln(y+1) - \ln(y-1)}{\frac{1}{y}} = \lim_{y \rightarrow \infty} \frac{\frac{1}{y+1} - \frac{1}{y-1}}{-\frac{1}{y^2}} = \lim_{y \rightarrow \infty} \frac{2y^2}{y^2 - 1} = 2.$$

3. $\lim_{x \rightarrow 0^-} x^3 e^{1/x}$

$$\lim_{x \rightarrow 0^-} x^3 e^{1/x} = 0 \cdot 0 = 0.$$

(L'Hôpital's Rule does not apply!)

4. $\lim_{x \rightarrow 0} \frac{x^2 + 3x}{\sinh x}$

$$\lim_{x \rightarrow 0} \frac{x^2 + 3x}{\sinh x} = \frac{0}{0} \text{ (L'Hôpital's Rule applies!)}$$

$$\lim_{x \rightarrow 0} \frac{x^2 + 3x}{\sinh x} = \lim_{x \rightarrow 0} \frac{2x + 3}{\cosh x} = \frac{3}{1} = 3.$$

5. $\lim_{x \rightarrow 1^+} \left(\frac{1}{\ln(1+x)} - \frac{1}{x} \right)$

$$\lim_{x \rightarrow 1^+} \left(\frac{1}{\ln(1+x)} - \frac{1}{x} \right) = \frac{1}{\ln 2} - 1.$$

(L'Hôpital's Rule does not apply!)

6. $\lim_{x \rightarrow \infty} (\ln |2x - 4| - \ln |x + 3|)$

$$\lim_{x \rightarrow \infty} (\ln |2x - 4| - \ln |x + 3|) = \infty - \infty \text{ (Indeterminate form - we need to rewrite!)}$$

$$\lim_{x \rightarrow \infty} (\ln |2x - 4| - \ln |x + 3|) = \lim_{x \rightarrow \infty} \ln \left| \frac{2x - 4}{x + 3} \right| = \ln \left| \frac{2}{1} \right| = \ln(2).$$

$$7. \lim_{\theta \rightarrow \infty} \theta \sin\left(\frac{1}{\theta}\right)$$

$$\lim_{\theta \rightarrow \infty} \theta \sin\left(\frac{1}{\theta}\right) = \infty \cdot 0 \text{ (Indeterminate form – we need to rewrite!)}$$

$$\lim_{\theta \rightarrow \infty} \theta \sin\left(\frac{1}{\theta}\right) = \lim_{\theta \rightarrow \infty} \frac{\sin\left(\frac{1}{\theta}\right)}{\frac{1}{\theta}} = \frac{0}{0} \text{ (L'Hôpital's Rule applies!)}$$

$$\lim_{\theta \rightarrow \infty} \theta \sin\left(\frac{1}{\theta}\right) = \lim_{\theta \rightarrow \infty} \frac{\sin\left(\frac{1}{\theta}\right)}{\frac{1}{\theta}} = \lim_{\theta \rightarrow \infty} \frac{\cos\left(\frac{1}{\theta}\right)\left(\frac{-1}{\theta^2}\right)}{\frac{-1}{\theta^2}} = \lim_{\theta \rightarrow \infty} \cos\left(\frac{1}{\theta}\right) = 1.$$

$$8. \lim_{y \rightarrow 0} \frac{2^y}{y^2}$$

$$\lim_{y \rightarrow 0} \frac{2^y}{y^2} = \frac{1}{0} = \infty.$$

(L'Hôpital's Rule does not apply!)

$$9. \text{ Use } \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e \text{ and substitution to find}$$

$$(a) \lim_{t \rightarrow 0^+} (1+t)^{1/t}$$

If we let $t = \frac{1}{x}$, then $\frac{1}{t} = x$. If we let $t \rightarrow 0^+$, then this means $x \rightarrow \infty$. Thus, we can rewrite our limit as:

$$\lim_{t \rightarrow 0^+} (1+t)^{1/t} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e.$$

$$(b) \lim_{y \rightarrow \infty} \left(1 + \frac{5}{y}\right)^y$$

If we let $t = \frac{5}{y}$, then $\frac{5}{t} = y$. If $y \rightarrow \infty$, then we must have $t \rightarrow 0^+$. Thus, we can rewrite our limit as:

$$\lim_{y \rightarrow \infty} \left(1 + \frac{5}{y}\right)^y = \lim_{t \rightarrow 0^+} (1+t)^{5/t} = \left(\lim_{t \rightarrow 0^+} (1+t)^{1/t}\right)^5 = e^5.$$