Section 4.7
Examples

We first would like to review the derivation of L'Hôpital's Rule (showing again how local linearization comes into play).

Suppose $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$. What can we say about $\lim_{x\to a} \frac{f(x)}{g(x)}$?

If $f(x)$ and $g(x)$ are both differentiable functions, we can use local linearity to write

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)(x-a)+f(a)}{g'(a)(x-a)+g(a)}$$

for x values near a. This means,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(a)(x-a)+f(a)}{g'(a)(x-a)+g(a)} = \lim_{x\to a}\frac{f'(a)(x-a)+0}{g'(a)(x-a)+0} = \lim_{x\to a}\frac{f'(a)}{g'(a)} = \frac{f'(a)}{g'(a)}.$$

One can then generalize this to obtain L'Hôpital's Rule,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

In Problems 1 through 5, verify that L'Hôpital's Rule applies, and then use it to find the limit. The problems are grouped according to their type.

1. (a) $\lim_{x\to -1}\frac{x^2-1}{\ln(3x+4)}$

   (b) $\lim_{x\to 1}\frac{1-x+\ln x}{1+\cos(\pi x)}$

   (c) $\lim_{x\to \pi^+}\frac{\sin(3x)}{(x-\pi)^2}$

2. (a) $\lim_{t\to 0}\frac{t^2}{e^t}$

   (b) $\lim_{x\to \infty}\frac{3(\ln x)^2}{x}$

3. $\lim_{y\to\infty}\left(\frac{1}{y}-\frac{1}{e^y-1}\right)$

4. $\lim_{\theta\to\frac{\pi}{4}}(1-\tan\theta)\sec\left(\theta+\frac{\pi}{4}\right)$

5. (a) $\lim_{h\to 0}(1+h)^{1/h}$

   (b) $\lim_{z\to\infty}(1+z^2)^{1/\ln z}$

6. Explain why L'Hôpital's rule does not apply to the following limits:

   (a) $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\ln x}\right)$

   (b) $\lim_{x\to 0^-}\frac{\ln|x|}{x}$

   (c) $\lim_{h\to 1}\frac{h-1}{h^2+1}$

   (d) $\lim_{t\to\infty}\left(\ln\left(\frac{1}{t}\right)-e^t\right)$

Solutions.

In Problems 1 through 5, verify that L'Hôpital's Rule applies, and then use it to find the limit. The problems are grouped according to their type.

1. (a) $\lim_{x\to -1} \dfrac{x^2-1}{\ln(3x+4)}$

$$\lim_{x\to -1} \frac{x^2-1}{\ln(3x+4)} = \frac{0}{\ln 7} = 0.$$

(L'Hôpital's Rule does not apply!)

(b) $\lim_{x\to 1} \dfrac{1-x+\ln x}{1+\cos(\pi x)}$

$$\begin{aligned}
\lim_{x\to 1} \frac{1-x+\ln x}{1+\cos(\pi x)} &= \frac{0}{0} \text{ (L'Hôpital's Rule applies!)}\\
\lim_{x\to 1} \frac{1-x+\ln x}{1+\cos(\pi x)} &= \lim_{x\to 1} \frac{-1+\frac{1}{x}}{-\pi\sin(\pi x)} = \frac{0}{0} \text{ (L'Hôpital's Rule applies!)}\\
\lim_{x\to 1} \frac{1-x+\ln x}{1+\cos(\pi x)} &= \lim_{x\to 1} \frac{-1+\frac{1}{x}}{-\pi\sin(\pi x)} = \lim_{x\to 1} \frac{-\frac{1}{x^2}}{-\pi^2\cos(\pi x)} = -\frac{1}{\pi^2}.
\end{aligned}$$

(c) $\lim_{x\to \pi^+} \dfrac{\sin(3x)}{(x-\pi)^2}$

$$\begin{aligned}
\lim_{x\to \pi^+} \frac{\sin(3x)}{(x-\pi)^2} &= \frac{0}{0} \text{ (L'Hôpital's Rule applies!)}\\
\lim_{x\to \pi^+} \frac{\sin(3x)}{(x-\pi)^2} &= \lim_{x\to \pi^+} \frac{3\cos(3x)}{2(x-\pi)} = \frac{-3}{0} = -\infty.
\end{aligned}$$

We note here that the above limit doesn't actual equal $\frac{-3}{0}$, since this is undefined – when we "compute" this limit and arrive at $\frac{-3}{0}$ (similar to when we arrive at $\frac{0}{0}$) we need to recognize that our work is not yet done. We need to interpret our $\frac{-3}{0}$ as a "big" number, -3, divided by a very small number (like 0.000001). If we divide a "big" number by a very small number, we get a HUGE number! So, if we divide -3 by a very small number (something close to 0) we will obtain a very big negative number – tending to $-\infty$, which is the true value of our limit.

2. (a) $\lim_{t\to 0} \dfrac{t^2}{e^t}$

$$\lim_{t\to 0} \frac{t^2}{e^t} = \frac{0}{1} = 0.$$

(L'Hôpital's Rule does not apply!)

(b) $\lim_{x\to \infty} \dfrac{3(\ln x)^2}{x}$

$$\begin{aligned}
\lim_{x\to \infty} \frac{3(\ln x)^2}{x} &= \frac{\infty}{\infty} \text{ (L'Hôpital's Rule applies!)}\\
\lim_{x\to \infty} \frac{3(\ln x)^2}{x} &= \lim_{x\to \infty} \frac{6(\ln x)\left(\frac{1}{x}\right)}{1} = \lim_{x\to \infty} \frac{6\ln x}{x} = \frac{\infty}{\infty} \text{ (L'Hôpital's Rule applies!)}\\
\lim_{x\to \infty} \frac{3(\ln x)^2}{x} &= \lim_{x\to \infty} \frac{6\ln x}{x} = \lim_{x\to \infty} \frac{6\left(\frac{1}{x}\right)}{1} = 0.
\end{aligned}$$

3. $\lim_{y\to\infty}\left(\frac{1}{y}-\frac{1}{e^y-1}\right)$

$$\lim_{y\to\infty}\left(\frac{1}{y}-\frac{1}{e^y-1}\right)=\infty-\infty \text{ (Indeterminate form – we need to rewrite!)}$$

$$\begin{aligned}
\lim_{y\to\infty}\left(\frac{1}{y}-\frac{1}{e^y-1}\right) &= \lim_{y\to 0^+}\frac{e^y-1-y}{y(e^y-1)}=\frac{0}{0} \text{ (L'Hôpital's Rule applies!)}\\
\lim_{y\to\infty}\left(\frac{1}{y}-\frac{1}{e^y-1}\right) &= \lim_{y\to 0^+}\frac{e^y-1-y}{y(e^y-1)}=\lim_{y\to 0^+}\frac{e^y-1}{ye^y+e^y-1}=\frac{0}{0} \text{ (L'Hôpital's Rule applies!)}\\
\lim_{y\to\infty}\left(\frac{1}{y}-\frac{1}{e^y-1}\right) &= \lim_{y\to 0^+}\frac{e^y-1-y}{y(e^y-1)}=\lim_{y\to 0^+}\frac{e^y-1}{ye^y+e^y-1}=\lim_{y\to 0^+}\frac{e^y}{ye^y+2e^y}=\frac{1}{2}.
\end{aligned}$$

4. $\lim_{\theta\to\frac{\pi}{4}}(1-\tan\theta)\sec\left(\theta+\frac{\pi}{4}\right)$

$$\lim_{\theta\to\frac{\pi}{4}}(1-\tan\theta)\sec\left(\theta+\frac{\pi}{4}\right)=0\cdot\infty \text{ (Indeterminate form – we need to rewrite!)}$$

$$\begin{aligned}
\lim_{\theta\to\frac{\pi}{4}}(1-\tan\theta)\sec\left(\theta+\frac{\pi}{4}\right) &= \lim_{\theta\to\frac{\pi}{4}}\frac{1-\tan\theta}{\cos\left(\theta+\frac{\pi}{4}\right)}=\frac{0}{0} \text{ (L'Hôpital's Rule applies!)}\\
\lim_{\theta\to\frac{\pi}{4}}(1-\tan\theta)\sec\left(\theta+\frac{\pi}{4}\right) &= \lim_{\theta\to\frac{\pi}{4}}\frac{1-\tan\theta}{\cos\left(\theta+\frac{\pi}{4}\right)}=\lim_{\theta\to\frac{\pi}{4}}\frac{-\sec^2\theta}{-\sin\left(\theta+\frac{\pi}{4}\right)}=\frac{2}{1}=2.
\end{aligned}$$

5. (a) $\lim_{h\to 0}(1+h)^{1/h}$

$$\lim_{h\to 0}(1+h)^{1/h}=1^\infty \text{ (Indeterminate form – we need to rewrite!)}$$

First, we note

$$\begin{aligned}
y &= (1+h)^{1/h}\\
\ln y &= \frac{\ln(1+h)}{h}.
\end{aligned}$$

Taking the limit of the right-side gives:

$$\begin{aligned}
\lim_{h\to 0}\frac{\ln(1+h)}{h} &= \frac{0}{0} \text{ (L'Hôpital's Rule applies!)}\\
\lim_{h\to 0}\frac{\ln(1+h)}{h} &= \lim_{h\to 0}\frac{\frac{1}{1+h}}{1}=1.
\end{aligned}$$

Thus,

$$\lim_{h\to 0}\ln y=\lim_{h\to 0}\frac{\ln(1+h)}{h}=1,$$

and so after exponentiating,

$$\lim_{h\to 0}y=e.$$

Finally, we can say,

$$\lim_{h\to 0}(1+h)^{1/h}=e.$$

(b) $\lim_{z\to\infty}(1+z^2)^{1/\ln z}$

$$\lim_{z\to\infty}(1+z^2)^{1/\ln z}=\infty^0 \text{ (Indeterminate form – we need to rewrite!)}$$

First, we note

$$\begin{aligned} y &= (1+z^2)^{1/\ln z} \\ \ln y &= \frac{\ln(1+z^2)}{\ln z}. \end{aligned}$$

Taking the limit of the right-side gives:

$$\begin{aligned} \lim_{z\to\infty} \frac{\ln(1+z^2)}{\ln z} &= \frac{\infty}{\infty} \text{ (L'Hôpital's Rule applies!)} \\ \lim_{z\to\infty} \frac{\ln(1+z^2)}{\ln z} &= \lim_{z\to\infty} \frac{\frac{2z}{1+z^2}}{\frac{1}{z}} = \lim_{z\to\infty} \frac{2z^2}{1+z^2} = \frac{2}{1} = 2. \end{aligned}$$

Thus,

$$\lim_{z\to\infty} \ln y = \lim_{z\to\infty} \frac{\ln(1+z^2)}{\ln z} = 2,$$

and so after exponentiating,

$$\lim_{h\to 0} y = e^2.$$

Finally, we can say,

$$\lim_{h\to 0} (1+z^2)^{1/\ln z} = e^2.$$

6. Explain why L'Hôpital's rule does not apply to the following limits:

(a) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\ln x} \right)$

$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\ln x} \right) = \infty - (-\infty) = \infty + \infty = \infty.$$

(b) $\lim_{x\to 0^-} \frac{\ln |x|}{x}$

$$\lim_{x\to 0^-} \frac{\ln |x|}{x} = \frac{-\infty}{0} = -\infty.$$

(c) $\lim_{h\to 1} \frac{h-1}{h^2+1}$

$$\lim_{h\to 1} \frac{h-1}{h^2+1} = \frac{0}{2} = 0.$$

(d) $\lim_{t\to\infty} \left(\ln\left(\frac{1}{t}\right) - e^t \right)$

$$\lim_{t\to\infty} \left(\ln\left(\frac{1}{t}\right) - e^t \right) = -\infty - (\infty) = -\infty - \infty = -\infty.$$