Section 4.6

Examples

1. (a) What is the volume formula for a sphere?
   (b) How fast does the volume of a spherical balloon change with respect to its radius?
   (c) How fast does the volume of the balloon change with respect to time?
   (d) If the radius is increasing at a constant rate of 0.03 inches per minute, how fast is the volume of the balloon changing at the time when its radius is 5 inches?

2. (a) What is the surface area formula for a closed cylinder?
   (b) If the radius of a cylinder is always $\frac{1}{4}$ of the height of the cylinder, how fast does the surface area of the cylinder change with respect to its height? With respect to its radius?
   (c) How fast does the surface area of a cylinder change with respect to time (assuming both radius and height change with respect to time)?
   (d) The surface area of a cylinder is increasing by 2π square inches per hour and the height is decreasing by 0.1 inches per hour when the radius is 16 inches and the height is 7 inches. How fast is the radius of the cylinder changing?

3. A searchlight is positioned 10 meters from a sidewalk. A person is walking along the sidewalk at a constant speed to 2 meters per second. The searchlight rotates so that it shines on the person. Find the rate at which the searchlight rotates when the person is 25 meters from the searchlight.

4. The length of a rectangle increases by 3 feet per minute while the width decreases by 2 feet per minute. When the length is 15 feet and the width is 40 feet, what is the rate at which the following change:
   (a) Area
   (b) Perimeter
   (c) Diagonal

Solutions.

1. (a) What is the volume formula for a sphere?

$$V = \frac{4}{3}\pi r^3$$

(b) How fast does the volume of a spherical balloon change with respect to its radius?

$$\frac{dV}{dr} = 4\pi r^2$$

(c) How fast does the volume of the balloon change with respect to time?

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

(d) If the radius is increasing at a constant rate of 0.03 inches per minute, how fast is the volume of the balloon changing at the time when its radius is 5 inches?

$$\left.\frac{dV}{dt}\right|_{r=5} = 4\pi(5)^2(0.03) = 3\pi \text{ in}^3/\text{min}$$

2. (a) What is the surface area formula for a closed cylinder?

$$S = 2\pi r^2 + 2\pi rh$$

(b) If the radius of a cylinder is always $\frac{1}{4}$ of the height of the cylinder, how fast does the surface area of the cylinder change with respect to its height? With respect to its radius?

With respect to height:
When $r = \frac{1}{4}h$,

$$\begin{aligned} S(h) &= 2\pi \left(\frac{1}{4}h\right)^2 + 2\pi \left(\frac{1}{4}h\right) h = \frac{5}{8}\pi h^2 \\ \frac{dS}{dh} &= \frac{5}{4}\pi h \end{aligned}$$

With respect to radius:
When $h = 4r$,

$$\begin{aligned} S(r) &= 2\pi r^2 + 2\pi r(4r) = 10\pi r^2 \\ \frac{dS}{dr} &= 20\pi r \end{aligned}$$

(c) How fast does the surface area of a cylinder change with respect to time (assuming both radius and height change with respect to time)?

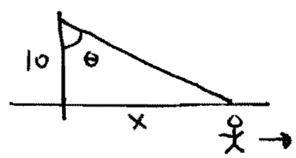
$$\frac{dS}{dt} = 4\pi r\frac{dr}{dt} + 2\pi r\frac{dh}{dt} + 2\pi h\frac{dr}{dt}$$

(d) The surface area of a cylinder is increasing by 2π square inches per hour and the height is decreasing by 0.1 inches per hour when the radius is 16 inches and the height is 7 inches. How fast is the radius of the cylinder changing?

$$\begin{aligned} 2\pi &= 4\pi(16)\frac{dr}{dt} + 2\pi(16)(-0.1) + 2\pi(7)\frac{dr}{dt} \\ 1 &= 32\frac{dr}{dt} - 1.6 + 7\frac{dr}{dt} \\ \frac{dr}{dt} &= \frac{2.6}{39} = \frac{1}{15} \text{ in/hr.} \end{aligned}$$

3. A searchlight is positioned 10 meters from a sidewalk. A person is walking along the sidewalk at a constant speed to 2 meters per second. The searchlight rotates so that it shines on the person. Find the rate at which the searchlight rotates when the person is 25 meters from the searchlight.

Sketching this scenario we see



$$\begin{aligned} \tan\theta &= \frac{x}{10} \\ \frac{d}{dt}(\tan\theta) &= \frac{d}{dt}\left(\frac{x}{10}\right) \\ \sec^2\theta\frac{d\theta}{dt} &= \frac{1}{10}\frac{dx}{dt} \end{aligned}$$

Now, when the person is 25 meters from the searchlight (i.e., the hypotenuse in the sketch is 25), we have $\sec\theta = \frac{25}{10} = 2.5$. Thus,

$$\begin{aligned} (2.5)^2\frac{d\theta}{dt} &= \frac{1}{10}(2) \\ \frac{d\theta}{dt} &= 0.032 \text{ rad/sec.} \end{aligned}$$

4. The length of a rectangle increases by 3 feet per minute while the width decreases by 2 feet per minute. When the length is 15 feet and the width is 40 feet, what is the rate at which the following change:

(a) Area

$$\begin{aligned} A &= \ell w \\ \frac{d}{dt}(A) &= \frac{d}{dt}(\ell w) \\ \frac{dA}{dt} &= \ell\frac{dw}{dt} + w\frac{d\ell}{dt} \\ \frac{dA}{dt} &= 15(-1) + 40(3) \\ &= 90 \text{ ft}^2\text{/min.} \end{aligned}$$

(b) Perimeter

$$\begin{aligned}
P &= 2\ell + 2w \\
\frac{d}{dt}(P) &= \frac{d}{dt}(2\ell + 2w) \\
\frac{dP}{dt} &= 2\frac{d\ell}{dt} + 2\frac{dw}{dt} \\
\frac{dP}{dt} &= 2(3) + 2(-2) = 2 \text{ ft/min.}
\end{aligned}$$

(c) Diagonal

$$\begin{aligned}
D^2 &= \ell^2 + w^2 \\
\frac{d}{dt}\left(D^2\right) &= \frac{d}{dt}\left(\ell^2 + w^2\right) \\
2D\frac{dD}{dt} &= 2\ell\frac{d\ell}{dt} + 2w\frac{dw}{dt} \\
D\frac{dD}{dt} &= \ell\frac{d\ell}{dt} + 2\frac{dw}{dt} \\
5\sqrt{73}\frac{dD}{dt} &= 15(3) + 40(-2) \\
\frac{dD}{dt} &= \frac{-7}{\sqrt{73}} \text{ ft/min.}
\end{aligned}$$