Section 4.3

Example

The following example illustrates the type of problem you will see in this section – we want to optimize some quantity, but we have a constraint.

1. We need to build a cylindrical fish tank. The bottom is made of slate and cost \$8 per square inch. The tube of glass can be purchased in any dimensions and costs \$3 per square inch. If the tank must hold 500 cubic inches, determine the dimensions that will minimize cost.

In this problem we want to optimize the total cost:

$$\text{Total Cost} = \text{Cost of Slate} + \text{Cost of Glass}$$

Cost is the product of cost per square inch and the number of inches needed:

$$\text{Total Cost} = (\text{Cost per square inch})\,(\text{Area of Circle}) + (\text{Cost per square inch})\,(\text{Area of Cylinder})\,.$$

The slate is in the shape of a circle, and the glass is in the shape of a cylinder:

$$\begin{aligned} \text{Area of Circle} &= \pi r^2 \\ \text{(Lateral surface) Area of a cylinder} &= 2\pi r h. \end{aligned}$$

Putting this together gives us a cost function in terms of radius and height:

$$C = 8(\pi r^2) + 3(2\pi r h).$$

We are not yet ready to compute a derivative because there are two variables in the right-hand side of our equation. The constraint in our problem will help us obtain a relationship between the two variables (and thus eliminate one).

If the tank must hold 500 cubic inches, our radius and height must satisfy the formula for the volume of a cylinder:

$$500 = \pi r^2 h.$$

We can rewrite this equation so that either the radius is expressed in terms of height, or vice-versa. Because height is the simplest to solve for, we should choose

$$h = \frac{500}{\pi r^2}$$

(Note, you should usually solve for the variable that isn't raised to a power – it will make your formulas easier to work with.)

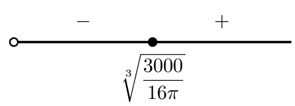
If we substitute this expression for h into our cost function, we obtain a function in terms only a single variable! (Hence we can take its derivative!)

$$\begin{aligned} C &= 8(\pi r^2) + 3(2\pi r h) \\ C(r) &= 8\pi r^2 + 6\pi r\left(\frac{500}{\pi r^2}\right) \\ &= 8\pi r^2 + 3000 r^{-1} \qquad \text{where } r > 0. \end{aligned}$$

Now, we are left with a problem similar to those from Section 4.2 – we just need to find critical points and test them.

$$\begin{aligned} C'(r) &= 16\pi r - 3000r^{-2} \\ 0 &= 16\pi r - \frac{3000}{r^2} \\ 0 &= \frac{16\pi r^3 - 3000}{r^2} \\ 0 &= 16\pi r^3 - 3000 \\ r &= \sqrt[3]{\frac{3000}{16\pi}}. \end{aligned}$$

Using the First Derivative Test, we see



The critical point is a minimum of the cost function because the sign of the first derivative changes from negative to positive.

If we use the Second Derivative Test,

$$\begin{aligned} C''(r) &= 16\pi + 6000r^{-3} \\ C''\left(\sqrt[3]{\frac{3000}{16\pi}}\right) &= 16\pi + 6000\left(\sqrt[3]{\frac{3000}{16\pi}}\right)^{-3} = 48\pi > 0. \end{aligned}$$

The critical point is a minimum because the cost function is concave up.

Thus, we know $r = \sqrt[3]{\frac{3000}{16\pi}}$ will minimize total cost, but the problem asks for the **dimensions** not just the radius – so we need to find h.

From above we know that

$$h = \frac{500}{\pi r^2}$$

so using $r = \sqrt[3]{\frac{3000}{16\pi}}$ we find,

$$h = \frac{500}{\pi\left(\sqrt[3]{\frac{3000}{16\pi}}\right)^2}.$$

Note, we *could* simplify h further to find that

$$h = \frac{20}{\pi}\left(\frac{2}{3\pi}\right)^{2/3},$$

but this simplification is rather difficult to come-by, so the previous h-value would be sufficient in this problem.