Section 4.2

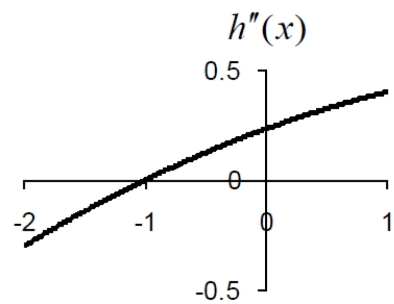
Examples

1. $f(x) = \left(\frac{\ln(x)}{x}\right)^2$
   (a) Find the derivative and use it to find all critical points.
   (b) Find the global maximum and global minimum (if possible) on each of the following intervals:
      i. $\left[\frac{1}{2}, 5\right]$
      ii. $[5, 10]$
      iii. $(0, \infty)$

2. $g(t) = t\sqrt{t+1}$
   (a) Find the derivative and use it to find all critical points.
   (b) Find the global maximum and global minimum (if possible) on the interval $[-1, 0]$.

3. Let $f(x) = At^2e^{-Bt}$ for $t \geq 0$ and $B > 0$. Find the local and global extrema.

4. Consider the function $h(x)$ on the interval $[-2, 1]$ with the following characteristics: $h'(-1) = 0$, $h(-1) = 2$, and the graph of h'' is shown below:



   (a) Explain why $h'(x)$ can never be negative on the interval.
   (b) Explain why $h(x)$ has a global maximum at $x = 1$.
   (c) Sketch a possible graph of $h(x)$ on the interval.

Solutions.

1. $f(x) = \left(\frac{\ln(x)}{x}\right)^2$

   (a) Find the derivative and use it to find all critical points.

   We first note that the domain is $(0, \infty)$ with a vertical asymptote at $x = 0$.

$$f'(x) = 2\left(\frac{\ln(x)}{x}\right)\left(\frac{x\frac{1}{x} - \ln(x)}{x^2}\right) = 2\left(\frac{\ln(x)(1 - \ln(x))}{x^3}\right).$$

   Critical Points: $x = 1$ and $x = e$.

   (b) Find the global maximum and global minimum (if possible) on each of the following intervals:

   i. $\left[\frac{1}{2}, 5\right]$

$$f\left(\frac{1}{2}\right) = \left(\frac{\ln\left(\frac{1}{2}\right)}{\frac{1}{2}}\right)^2 \approx 1.92 \qquad f(1) = 0 \qquad f(e) = \frac{1}{e^2} \approx 0.14 \qquad f(5) = \left(\frac{\ln(5)}{5}\right)^2 \approx 0.10.$$

   Global Maximum at $x = \frac{1}{2}$ Global Minimum at $x = 1$.

   ii. $[5, 10]$

$$f(5) = \left(\frac{\ln(5)}{5}\right)^2 \approx 0.10 \qquad f(10) = \left(\frac{\ln(10)}{10}\right)^2 \approx 0.05 \qquad \text{No critical points in } [5, 10]$$

   Global Maximum at $x = 5$ Global Minimum at $x = 10$.

   iii. $(0, \infty)$

   Global Maximum: None Global Minimum at $x = 1$.

2. $g(t) = t\sqrt{t+1}$

   (a) Find the derivative and use it to find all critical points.

$$\begin{aligned} g'(t) &= t\left(\frac{1}{2}\right)(t+1)^{-1/2} + (t+1)^{1/2} \\ &= \frac{1}{2(t+1)^{1/2}} + (t+1)^{1/2} \\ &= \frac{t + 2(t+1)}{2(t+1)^{1/2}} \\ &= \frac{3t+2}{2(t+1)^{1/2}} \end{aligned}$$

   Critical Points: $t = -\frac{2}{3}$ and $t = -1$.

(b) Find the global maximum and global minimum (if possible) on the interval $[-1, 0]$.

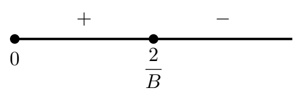
$$g(-1) = 0 \qquad g\left(-\frac{2}{3}\right) = -\frac{2}{3}\sqrt{\frac{1}{3}} \approx -0.385 \qquad g(0) = 0.$$

Global Minimum at $t = -\frac{2}{3}$ Global Maximum at $t = -1$ and $t = 0$.

3. Let $f(x) = At^2 e^{-Bt}$ for $t \geq 0$ and $B > 0$. Find the local and global extrema.

$$f'(t) = At^2\left(-Be^{-Bt}\right) + e^{-Bt}(2At) = Ate^{-Bt}(-Bt + 2).$$

Critial Points: $t = 0$ and $t = \frac{2}{B}$.

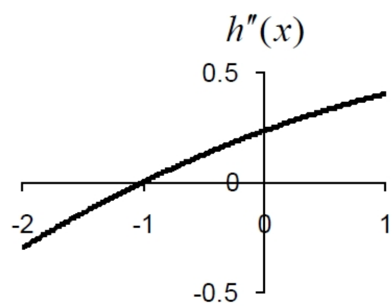


$\lim_{t\to\infty} f(t) = 0 \qquad f(0) = 0.$

$t = \frac{2}{B}$ is the location of both a local and global maximum.

$t = 0$ is the location of a global minimum

4. Consider the function $h(x)$ on the interval $[-2, 1]$ with the following characteristics: $h'(-1) = 0$, $h(-1) = 2$, and the graph of h'' is shown below:



(a) Explain why $h'(x)$ can never be negative on the interval.

$$h''(-1) = 0$$
$$h''(x) < 0 \text{ for } x < -1$$
$$h''(x) > 0 \text{ for } x > -1$$

These tell us that $x = -1$ is a local minimum on the graph of $h'(x)$.

Because $h'(-1) = 0$, the y-values on $h'(x)$ must be greater than or equal to 0.

(b) Explain why $h(x)$ has a global maximum at $x = 1$.

Since $h'(x)$ is not negative on the interval (as shown above), $h(x)$ is increasing on the interval. The largest y-value of $h(x)$ will then occur at the right endpoint of the interval, i.e., at $x = 1$.

(c) Sketch a possible graph of $h(x)$ on the interval.