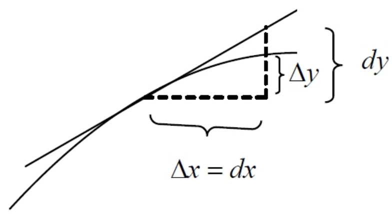
Section 3.9

Examples

Consider a function $y = f(x)$ and a point on its graph at $x = a$.

Tangent line to the graph of f :	$y = f'(a)(x - a) + f(a)$
Local linearization of f :	$f'(a)(x - a) + f(a)$
Differential of f :	$dy = f'(x)dx$
Approximation of changes in f :	$\Delta y \approx f'(x)\Delta x$
Relative or % changes in f :	$\dfrac{\Delta y}{y}$



1. Suppose $f'(x)$ is a differentiable, decreasing function for all x. Which is larger, $f(5+\Delta x)$ or $f(5)+f'(5)\Delta x$?

2. Find the equation of the tangent line to $y = \cos(x)$ at $x = \frac{\pi}{4}$. If the tangent line is used to approximate values of $y = \cos(x)$ over the interval $0 \leq x \leq \frac{\pi}{2}$, would they be under approximations or over approximations?

3. Use the linearization of $y = (1 + x)^k$ to estimate $\sqrt[4]{1.1}$.

4. Consider the volume of a sphere.
   (a) If the radius is measured as 5 inches with an approximate error of ± 0.05 inches, what is an approximate error in calculating the volume?
   (b) If the relative error in the radius is approximately 2%, what is the relative error in calculating the volume?

5. The period, T, of a pendulum of length L is given by the equation below, where g is the acceleration constant due to gravity. If the length of the pendulum changes by ΔL, how will the period change?

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

Solutions.

Consider a function $y = f(x)$ and a point on its graph at $x = a$.

Tangent line to the graph of f :	$y = f'(a)(x-a) + f(a)$
Local linearization of f :	$f'(a)(x-a) + f(a)$
Differential of f :	$dy = f'(x)dx$
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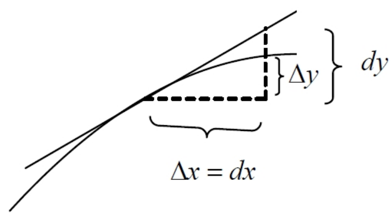
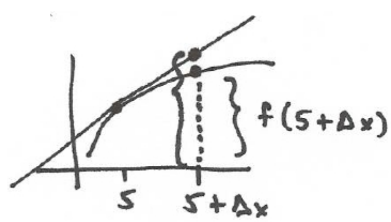


1. Suppose $f'(x)$ is a differentiable, decreasing function for all x. Which is larger, $f(5+\Delta x)$ or $f(5)+f'(5)\Delta x$?

   First, note that $f(5 + \Delta x)$ lies on the graph of f (it is the function evaluated at a particular point) while $f(5) + f'(5)\Delta x$ lies on the graph of the tangent line (since the tangent line has the form $y = f'(5)(x - 5) + f(5)$).

   Thus we can see that $f(5) + f'(5)\Delta x$ is the larger value.

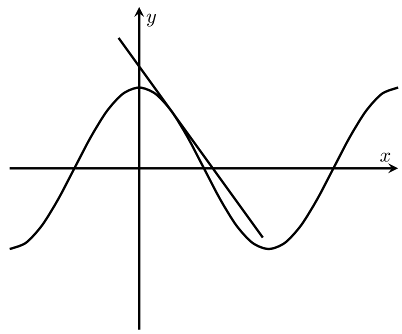
2. Find the equation of the tangent line to $y = \cos(x)$ at $x = \frac{\pi}{4}$. If the tangent line is used to approximate values of $y = \cos(x)$ over the interval $0 \le x \le \frac{\pi}{2}$, would they be under approximations or over approximations?

$$f(x) = \cos(x) \qquad f\left(\tfrac{\pi}{4}\right) = \tfrac{\sqrt{2}}{2}$$
$$f'(x) = -\sin(x) \qquad f'\left(\tfrac{\pi}{4}\right) = -\tfrac{\sqrt{2}}{2}$$

   So our tangent line is given by

$$y = -\frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}.$$

Sketching the graphs,



we see that the tangent line would be an over approximation.

3. Use the linearization of $y = (1+x)^k$ to estimate $\sqrt[4]{1.1}$.

$$\begin{array}{ll} y = (1+x)^k & y(0) = 1 \\ y' = k(1+x)^{k-1} & y'(0) = \frac{1}{4} \end{array}$$

To estimate $\sqrt[4]{1.1}$ we need to take $k = \frac{1}{4}$ (since $\sqrt[4]{1.1} = (1+0.1)^{1/4}$). We now see

$$\begin{aligned} (1+x)^{1/4} &\approx \frac{1}{4}(x-0)+1 \\ (1.1)^{1/4} &\approx \frac{1}{4}(0.1+0)+1 = 1.025. \end{aligned}$$

4. Consider the volume of a sphere.

(a) If the radius is measured as 5 inches with an approximate error of ± 0.05 inches, what is an approximate error in calculating the volume?

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ \frac{dV}{dr} &= 4\pi r^2 \\ dV &= 4\pi r^2 dr \\ \Delta V &\approx 4\pi r^2 \Delta r \\ \Delta V &\approx 4\pi (5)^2(\pm 0.05) = \pm 5\pi \text{ in}^3. \end{aligned}$$

(b) If the relative error in the radius is approximately 2%, what is the relative error in calculating the volume?

$$\begin{aligned} dV &= 4\pi r^2 dr \\ \frac{dV}{V} &= \frac{4\pi r^2 dr}{V} \\ \frac{dV}{V} &= \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} \\ \frac{dV}{V} &= 3\frac{dr}{r} \\ \frac{\Delta V}{V} &\approx 3\frac{\Delta r}{r} \\ \frac{\Delta V}{V} &\approx 3(0.02) = 0.06 = 6\%. \end{aligned}$$

5. The period, T, of a pendulum of length L is given by the equation below, where g is the acceleration constant due to gravity. If the length of the pendulum changes by ΔL, how will the period change?

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

We first note, that using a bit a algebra, $\dfrac{2\pi}{\sqrt{g}} = \dfrac{T}{\sqrt{L}}$ (this will come in handy later).

Now,

$$\begin{aligned}
\frac{dT}{dL} &= \frac{2\pi}{\sqrt{g}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{L}} \\
dT &= \frac{2\pi}{\sqrt{g}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{L}} dL \\
dT &= \frac{T}{\sqrt{L}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{L}} dL \\
dT &= \frac{1}{2}\frac{T}{L} dL \\
\Delta T &\approx \frac{1}{2}\frac{T}{L} \Delta L.
\end{aligned}$$