Section 3.7

Examples

1. Which of the following equations represents a function of x, only a relation, or nonsense?
   (a) $\cos(xy) = x^2 + y^2$
   (b) $e^x y + 4x^2 = 7$
   (c) $x^2 + y^2 + 3 = 0$
   (d) $y = \ln(\ln(\sin(x)))$

2. Consider $x^3 + 4y^2 = 3$.
   (a) Find $\frac{dy}{dx}$.
   (b) Find points where the tangent line is vertical. Is horizontal.
   (c) Find the equation of the tangent line at $(-1, 1)$.

3. Find $\frac{dy}{dx}$ for $3^x + y^2 \ln(x^3 + 1) = 4y^2 + 10$.

4. Find $\frac{dz}{dt}$ for $6tz^3 - 4t = 7e^z$

5. Find $\frac{d\theta}{dt}$ for $\tan^2(\theta) = x^3 + 1$.

6. Find $f'(x)$ for $f(x) = x^x$.

Solutions.

1. Which of the following equations represents a function of x, only a relation, or nonsense?

   (a) $\cos(xy) = x^2 + y^2$

   - Relation

   (b) $e^x y + 4x^2 = 7$

   - Function of x : $y = \dfrac{7-4x^2}{e^x}$

   (c) $x^2 + y^2 + 3 = 0$

   - Nonsense : $x^2 + y^2 \neq -3$ (the left side is alway positive!)

   (d) $y = \ln(\ln(\sin(x)))$

   - Nonsense : The domain of $\ln(x)$ is $x > 0$, but there are x-values for which $\sin(x) \leq 0$ – this would make $\ln(\sin(x))$ undefined!

2. Consider $x^3 + 4y^2 = 3$.

   (a) Find $\dfrac{dy}{dx}$.

$$\begin{aligned} 3x^2 + 8y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-3x^2}{8y}. \end{aligned}$$

   (b) Find points where the tangent line is vertical. Is horizontal.

   Vertical:

$$\begin{aligned} 8y &= 0 \\ y &= 0. \end{aligned}$$

   The point corresponding to $y = 0$ is: $(\sqrt[3]{3}, 0)$.

   Horizontal:

$$\begin{aligned} -3x^2 &= 0 \\ x &= 0. \end{aligned}$$

   The points corresponding to $x = 0$ are: $(0, \sqrt{\frac{3}{4}})$ and $(0, -\sqrt{\frac{3}{4}})$.

   (c) Find the equation of the tangent line at $(-1, 1)$.

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{-3(-1)^2}{8(1)} = -\frac{3}{8}$$

   So the tangent line is:

$$y = -\frac{3}{8}(x+1) + 1.$$

3. Find $\frac{dy}{dx}$ for $3^x + y^2 \ln(x^3+1) = 4y^2 + 10$.

$$\begin{aligned}
\ln(3)3^x + y^2 \cdot \frac{1}{x^3+1} \cdot 3x^2 + \ln(x^3+1) \cdot 2y\frac{dy}{dx} &= 8y\frac{dy}{dx} \\
\ln(3)3^x + \frac{3x^2y^2}{x^3+1} + 2y\ln(x^3+1)\frac{dy}{dx} &= 8y\frac{dy}{dx} \\
\ln(3)3^x + \frac{3x^2y^2}{x^3+1} &= 8y\frac{dy}{dx} - 2y\ln(x^3+1)\frac{dy}{dx} \\
\frac{\ln(3)3^x + \frac{3x^2y^2}{x^3+1}}{8y - 2y\ln(x^3+1)} &= \frac{dy}{dx} \\
\frac{\ln(3)3^x(x^3+1) + 3x^2y^2}{(8y - 2y\ln(x^3+1))(x^3+1)} &= \frac{dy}{dx}.
\end{aligned}$$

4. Find $\frac{dz}{dt}$ for $6tz^3 - 4t = 7e^z$

$$\begin{aligned}
6t \cdot 3z^2\frac{dz}{dt} + z^3 \cdot 6 - 4 &= 7e^z\frac{dz}{dt} \\
18tz^2\frac{dz}{dt} + 6z^3 - 4 &= 7e^z\frac{dz}{dt} \\
6z^3 - 4 &= 7e^z\frac{dz}{dt} - 18tz^2\frac{dz}{dt} \\
\frac{6z^3 - 4}{7e^z - 18tz^2} &= \frac{dz}{dt}.
\end{aligned}$$

5. Find $\frac{d\theta}{dt}$ for $\tan^2(\theta) = x^3 + 1$.

$$\begin{aligned}
2\tan(\theta)\sec^2(\theta)\frac{d\theta}{dt} &= 3x^2 \\
\frac{d\theta}{dt} &= \frac{3x^2}{2\tan(\theta)\sec^2(\theta)}.
\end{aligned}$$

6. Find $f'(x)$ for $f(x) = x^x$.

(See the video for Section 3.7 for more on logarithmic differentiation.)

$$\begin{aligned}
y &= x^x \\
\ln(y) &= \ln(x^x) \\
\ln(y) &= x\ln(x) \\
\frac{d}{dx}\left(\ln(y)\right) &= \frac{d}{dx}\left(x\ln(x)\right) \\
\frac{1}{y}\frac{dy}{dx} &= x\cdot\frac{1}{x} + \ln(x)\cdot 1 \\
\frac{1}{y}\frac{dy}{dx} &= 1 + \ln(x) \\
\frac{dy}{dx} &= y(1 + \ln(x)) \\
f'(x) &= x^x(1 + \ln(x)).
\end{aligned}$$