Section 3.6

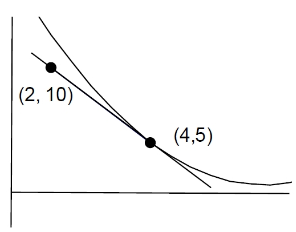
Examples

1. Find the first derivative for each function. Simplify your answers.
   (a) $f(p) = p^3 \ln(p^2 + 1)$
   (b) $h(t) = \dfrac{1}{\ln(t)}$
   (c) $p(z) = \ln\left(\dfrac{e^{3z} z^5}{(5z+3)^2}\right)$
   (d) $g(x) = \arcsin(x^3)$
   (e) $p(v) = \tan(\arctan(3v))$
   (f) $r(t) = \dfrac{1}{\tan^{-1}(t)}$

2. Determine where $f'(0) = 0$ for the function

$$f(x) = \begin{cases} \sin(\pi x) & -\infty < x < 1 \\ \ln(5x) & 1 \leq x < \infty. \end{cases}$$

3. A graph of $f(x)$ and its tangent line are shown below.



   Find the following values:
   (a) $h'(2)$ for $h(x) = f(x^2)$
   (b) $k'(4)$ for $k(x) = (f(x))^2$
   (c) $g'(5)$ for $g(x) = f^{-1}(x)$
   (d) $p'(4)$ for $p(x) = \ln(f(x))$

4. Suppose $f(t)$ models the number of motor vehicles (in millions) registered in the world since 1980. Give a practical interpretation for each of the following:
   (a) $f(20)$
   (b) $f^{-1}(400)$
   (c) $f'(20)$
   (d) $(f^{-1})'(400)$

Solutions.

1. Find the first derivative for each function. Simplify your answers.

(a) $f(p) = p^3 \ln(p^2+1)$

$$\begin{aligned} f'(p) &= p^3 \cdot \frac{1}{p^2+1} \cdot 2p + \ln(p^2+1) \cdot 3p^2 \\ &= \frac{2p^4}{p^3+1} + 3p^2 \ln(p^2+1). \end{aligned}$$

(b) $h(t) = \dfrac{1}{\ln(t)}$

$$\begin{aligned} h(t) &= (\ln(t))^{-1} \\ h'(t) &= -(\ln(t))^{-2} \cdot \frac{1}{t} \\ &= \frac{-1}{t\,(\ln(t))^2}. \end{aligned}$$

(c) $p(z) = \ln\left(\dfrac{e^{3z} z^5}{(5z+3)^2}\right)$

$$\begin{aligned} p(z) &= 3z + 5\ln(z) - 2\ln(5z+3) \\ p'(z) &= 3 + \frac{5}{z} - 2 \cdot \frac{1}{5z+3} \cdot 5 \\ &= 5 + \frac{5}{z} - \frac{10}{5z+3}. \end{aligned}$$

(d) $g(x) = \arcsin(x^3)$

$$\begin{aligned} g'(x) &= \frac{1}{\sqrt{1-(x^3)^2}} \cdot 3x^2 \\ &= \frac{3x^2}{\sqrt{1-x^6}}. \end{aligned}$$

(e) $p(v) = \tan(\arctan(3v))$
Provided the domain is appropriate, we may first simplify,

$$p(v) = 3v,$$

so that

$$p'(v) = 3.$$

(f) $r(t) = \dfrac{1}{\tan^{-1}(t)}$

$$\begin{aligned} r(t) &= \left(\tan^{-1}(t)\right)^{-1} \quad \text{or} \quad (\arctan(t))^{-1} \\ r'(t) &= -(\arctan(t))^{-2} \cdot \frac{1}{1+t^2} \\ &= \frac{-1}{(\arctan(t))^2(1+t^2)}. \end{aligned}$$

2. Determine where $f'(0) = 0$ for the function

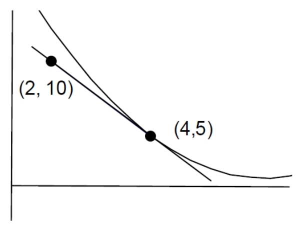
$$f(x) = \begin{cases} \sin(\pi x) & -\infty < x < 1 \\ \ln(5x) & 1 \le x < \infty. \end{cases}$$

First, since $f(x)$ is not continuous at $x = 1$, it is not differentiable at $x = 1$. Thus we have

$$f'(x) = \begin{cases} \pi\cos(\pi x) & -\infty < x < 1 \\ \frac{1}{x} & 1 < x < \infty. \end{cases}$$

From this we see that $f'(x) = 0$ only when $\pi\cos(\pi x) = 0$, or equivalently, when $\cos(\pi x) = 0$.

3. A graph of $f(x)$ and its tangent line are shown below.



Find the following values:

(a) $h'(2)$ for $h(x) = f(x^2)$

$$\begin{aligned} h'(x) &= f'(x^2) \cdot 2x \\ h'(2) &= f'(4) \cdot 2 \cdot 2 = -\frac{5}{2} \cdot 2 \cdot 2 = -10. \end{aligned}$$

(b) $k'(4)$ for $k(x) = (f(x))^2$

$$\begin{aligned} k'(x) &= 2f(x) \cdot f'(x) \\ k'(4) &= 2f(4) \cdot f'(4) = 2 \cdot 5 \cdot \left(-\frac{5}{2}\right) = -25. \end{aligned}$$

(c) $g'(5)$ for $g(x) = f^{-1}(x)$

$$\begin{aligned} g'(x) &= \frac{1}{f'(f^{-1}(x))} \\ g'(5) &= \frac{1}{f'(f^{-1}(5))} \\ &= \frac{1}{f'(4)} = \frac{1}{-\frac{5}{2}} = -\frac{2}{5}. \end{aligned}$$

(d) $p'(4)$ for $p(x) = \ln(f(x))$

$$\begin{aligned} p'(x) &= \frac{1}{f(x)} \cdot f'(x) \\ p'(4) &= \frac{1}{f(4)} \cdot f'(4) \\ &= \frac{1}{5} \cdot \left(-\frac{5}{2}\right) = -\frac{1}{2}. \end{aligned}$$

4. Suppose $f(t)$ models the number of motor vehicles (in millions) registered in the world since 1980. Give a practical interpretation for each of the following:

   (a) $f(20)$

   The number of motor vehicles (in millions) that were registered in the world in the year 2000.

   (b) $f^{-1}(400)$

   This is the number of hears after 1980 when there were 400 million motor vehicles registered in the world.

   (c) $f'(20)$

   Between 2000 and 2001, approximately $f'(20)$ million additional motor vehicles were registered in the world.

   (d) $(f^{-1})'(400)$

   When there were 400 million motor vehicles registered in the world, it took about $(f^{-1})'(400)$ years for another million vehicles to be registered.