Section 3.3

Examples

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1. Find all values of t so that $f(t) = e^t(t^3 - 4)$ has a horizontal tangent.

2. Revenue can be expressed as price per item multiplied by the number of times, where the number of items is a function of price: $R(p) = pq = p \cdot f(p)$. Suppose we sell a watch for \$60, $f(60) = 500$, and $f'(60) = -8$.
   (a) Give practical interpretations for $f(60) = 500$ and $f'(60) = -8$.
   (b) Find values and give practical interpretations for $R(60)$ and $R'(60)$.

3. Find the first derivative for each. Simplify your answers.
   (a) $f(x) = ax^2b^x$
   (b) $g(z) = \dfrac{az+b}{az-c}$
   (c) $h(t) = \dfrac{\sqrt[3]{t}}{1+4t^3}$
   (d) $p(t) = \left(\dfrac{t+4}{t-3}\right)(t^3 - 1)$
   (e) $y = \dfrac{e^r r^3}{2+r}$
   (f) $g(t) = \begin{cases} te^t & t \leq 0 \\ \dfrac{t}{t^2-1} & t > 0. \end{cases}$

4. Suppose f, g, and h are differentiable functions. Use the table of values to find the following.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	3	4	-1	5
3	-1	-2	4	2

   (a) $\left.\dfrac{dy}{dx}\right|_{x=2}$ for $y = 3f(x)g(x)$.
   (b) Determine if $y = \dfrac{g(x)}{f(x)}$ is increasing or decreasing at $x = 3$.

5. Suppose $f(v)$ is liters/km used, and $h(v)$ is liters/hr used when traveling v km/hr. If $h(80) = 0.4$, and $h'(80) = 0.09$, find the values, and give practical interpretations for $f(80)$ and $f'(80)$.

Solutions.

1. Find all values of t so that $f(t) = e^t(t^3 - 4)$ has a horizontal tangent.

$$\begin{aligned} f'(t) &= e^t \cdot et^2 + (t^3 - 4) \cdot e^t \\ &= 3t^2 e^t + (t^3 - 4)e^t. \end{aligned}$$

Now, solving $f'(t) = 0$ for t,

$$\begin{aligned} 0 &= 3t^2 e^t + (t^3 - 4)e^t \\ &= e^t(3t^2 + t^3 - 4) \\ &= e^t(t - 1)(t + 2)(t + 2), \end{aligned}$$

since $e^t \neq 0$, we have $t = 1$ and $t = -2$.

2. Revenue can be expressed as price per item multiplied by the number of times, where the number of items is a function of price: $R(p) = pq = p \cdot f(p)$. Suppose we sell a watch for \$60, $f(60) = 500$, and $f'(60) = -8$.

(a) Give practical interpretations for $f(60) = 500$ and $f'(60) = -8$.

$f(60) = 500 \Rightarrow$ If the price of the watch is \$60, we will sell 500 watches.

$f'(60) = -8 \Rightarrow$ If we increase the price of the watch from \$60 to \$61, we will sell 8 fewer watches.

(b) Find values and give practical interpretations for $R(60)$ and $R'(60)$.

$$R(60) = 60 \cdot 500 = 30,000$$

The revenue from selling \$60 watches is \$30,000.

$$\begin{aligned} R'(p) &= p \cdot f'(p) + f(p) \cdot 1 \\ R'(60) &= 60 \cdot f'(60) + f(60) \\ &= 60(-8) + 500 \\ &= 20. \end{aligned}$$

If we increase the price of the watch from \$60 to \$61, our revenue will increase by \$20.

3. Find the first derivative for each. Simplify your answers.

(a) $f(x) = ax^2 b^x$

$$\begin{aligned} f'(x) &= a\left(x^2 (\ln b)\, b^x + 2x \cdot b^x\right) \\ &= axb^x (x \ln b + 2). \end{aligned}$$

Note, we factored out common terms!

(b) $g(z) = \dfrac{az+b}{az-c}$

$$\begin{aligned} g'(z) &= \frac{(az-c)a - (az+b)a}{(az-c)^2} \\ &= \frac{a^2z - ac - a^2z - ab}{(az-c)^2} \\ &= \frac{-ac-ab}{(az-c)^2}. \end{aligned}$$

Note, we simplified the numerator, but did not distribute the denominator!

(c) $h(t) = \dfrac{\sqrt[3]{t}}{1+4t^3}$

$$\begin{aligned} h'(t) &= \frac{(1+4t^3)\frac{1}{3}t^{-2/3} - t^{1/3}12t^2}{(1+4t^3)^2} \\ &= \frac{(1+4t^3)\frac{1}{3}t^{-2/3} - t^{1/3}12t^2}{(1+4t^3)^2} \cdot \frac{3t^{2/3}}{3t^{2/3}} \\ &= \frac{1+4t^3-36t^3}{3t^{2/3}(1+4t^3)^2} \\ &= \frac{1-32t^3}{3t^{2/3}(1+4t^3)^2}. \end{aligned}$$

Note, we multiplied both numerator and denominator by $3t^{2/3}$ to make our simplification easier. We multiplied by this quantity since a term with a negative exponent appeared, i.e., since we saw $\frac{1}{3}t^{-2/3}$ (which is the same as $\frac{1}{3t^{2/3}}$) we multiply numerator and denominator by $3t^{2/3}$. (This is a handy trick for simplifying when you see negative powers in a quotient.)

(d) $p(t) = \left(\dfrac{t+4}{t-3}\right)(t^3-1)$

$$\begin{aligned} p'(t) &= \left(\frac{t+4}{t-3}\right)3t^2 + (t^3-1)\left(\frac{(t-3)\cdot 1 - (t+4)\cdot 1}{(t-3)^2}\right) \\ &= 3t^2\left(\frac{t+4}{t-3}\right) - \frac{7(t^3-1)}{(t-3)^2}. \end{aligned}$$

(e) $y = \dfrac{e^r r^3}{2+r}$

$$\begin{aligned} \frac{dy}{dr} &= \frac{(2+r)(e^r er^2 + r^3e^r) - e^r r^3 \cdot 1}{(2+r)^2} \\ &= \frac{(2+r)(3+r)r^2e^r - r^3e^r}{(2+r)^2}. \end{aligned}$$

Note, it is OK to leave your answer in this form – there are no “easy” simplifications or factoring you can do.

(f) $g(t) = \begin{cases} te^t & t \le 0 \\ \dfrac{t}{t^2-1} & t > 0. \end{cases}$

We first note that $g(t)$ is not differentiable at $t = 0$.

$$\begin{aligned} g'(t) &= \begin{cases} te^t + e^t & t < 0 \\ \frac{(t^2-1)\cdot 1 - t\cdot 2t}{(t^2-1)^2} & t > 0, \end{cases} \\ &= \begin{cases} e^t(t+1) & t < 0 \\ \frac{-1-t^2}{(t^2-1)^2} & t > 0. \end{cases} \end{aligned}$$

4. Suppose f, g, and h are differentiable functions. Use the table of values to find the following.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	3	4	-1	5
3	-1	-2	4	2

(a) $\left.\frac{dy}{dx}\right|_{x=2}$ for $y = 3f(x)g(x)$.

$$\begin{aligned} \frac{dy}{dx} &= 3\left(f(x)g'(x) + f'(x)g(x)\right) \\ \left.\frac{dy}{dx}\right|_{x=2} &= 3\left(f(2)g'(2) + f'(2)g(2)\right) = 33. \end{aligned}$$

(b) Determine if $y = \frac{g(x)}{f(x)}$ is increasing or decreasing at $x = 3$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{f(x)g'(x) - g(x)f'(x)}{f(x)^2} \\ \left.\frac{dy}{dx}\right|_{x=3} &= \frac{f(3)g'(3) - g(3)f'(3)}{f(3)^2} \\ &= \frac{(-1)(2) - (-2)(4)}{(-1)^2} = 6 > 0. \end{aligned}$$

Since $\left.\frac{dy}{dx}\right|_{x=3} > 0$, we know $y = \frac{g(x)}{f(x)}$ is increasing.

5. Suppose $f(v)$ is liters/km used, and $h(v)$ is liters/hr used when traveling v km/hr. If $h(80) = 0.4$, and $h'(80) = 0.09$, find the values, and give practical interpretations for $f(80)$ and $f'(80)$.

We begin by investigating the given units. Notice that,

$$\frac{\text{liters/hr}}{\text{km/hr}} = \text{liters/km},$$

which tells us that

$$f(v) = \frac{h(v)}{v}.$$

Now,

$$f(80) = \frac{0.4}{80} = 0.005,$$

which says that you are using 0.005 liters of gas per km when you are traveling 80 km/hr.

Also,

$$\begin{aligned} f'(v) &= \frac{vh'(v) - h(v)}{v^2} \\ f'(80) &= \frac{80(0.09) - 0.4}{80^2} \\ &= 0.00106, \end{aligned}$$

which says that if you travel 81 km/hr instead of 80 km/hr, your fuel usage will increase by approximately 0.001 liters per km.