# Categorization of all Newton maps of rational functions conjugate to quadratic polynomials

by

Erin Elizabeth Williams, B.S., M.S.

A Dissertation

In

Mathematics and Statistics

Submitted to the Graduate Faculty
of Texas Tech University in
Partial Fulfillment of
the Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Approved

Dr. Roger W. Barnard
Co-Chair

Dr. Jerry Dwyer
Co-Chair

Dr. Brock Williams

Dominick Casadonte
Interim Dean of the Graduate School

August, 2013

# ACKNOWLEDGMENTS

First I would like to thank my two advisors Dr. Roger W. Barnard and Dr. Jerry Dwyer for all of their support and guidance through writing this dissertation as well as throughout my time at Texas Tech. My committee member, Dr. Brock Williams, must also be thanked for all of his help through this process and my studies. I could not have been more blessed with a better committee than the one that I had, and I am forever grateful.

Additionally, I would like to thank those outside of my committee whose questions and input helped to strengthen this work. Specifically, Dr. Kent Pearce who was not officially on my committee, but his insightful questions and support improved my dissertation. In addition, both Dr. Phil Rippon and Dr. Alan Beardon answered questions through communication with my advisor Dr. Barnard. I also would like to thank Dr. Chris Monico for some nice conversations with further exploration ideas during the final few weeks.

Thanks also goes to Carol Williams for all of her support and friendship, especially during the past few years. My family for their love and encouragement throughout the years, during the times working on my dissertation and far prior to this.

Not enough thanks can be given to my husband Scottie, without his love, friendship and support this could not have been completed. Thank you.

Finally, I would like to thank God for all His help and providing me with confidence in my own work. There were times when His strength was all I needed.

## TABLE OF CONTENTS

# ABSTRACT

Newton's method is a well-known iterative method to find roots of a function. It has primarily been studied for polynomials, but more recently transcendental functions have also been studied. However, according to our literature search, nothing has been studied for Newton's method for rational functions. Here we extend the known properties of the degree and fixed points of Newton maps of polynomials to Newton maps of rational functions. We also categorize all Newton maps conjugate to quadratic polynomials and show the complications in extending this to degree three Newton maps.

# LIST OF FIGURES

# CHAPTER 1
# INTRODUCTION

The iteration of rational functions was first investigated by Pierre Fatou and Gaston Julia whose articles were published between 1918 and 1920 [20], [25]. During this time they had access to a theorem of Paul Montel:

**Theorem 1.1.1.** *(Montel) Let* $\mathscr{F}$ *be a family of meromorphic functions defined on a domain* $U$*. Suppose there exist points* $a, b, c \in \mathbb{C}_\infty$ *such that* $\left\{\bigcup_{f\in\mathscr{F}} f(U)\right\} \cap \{a, b, c\} = \emptyset$*. Then* $\mathscr{F}$ *is a normal family on* $U$*.*

Using this result they applied normal families to the theory of iteration to produce fascinating results (see § 2.7 for definition of normality, and [20], [25] for these results). Joseph Ritt also published an article in 1920 which included "work which has not been covered by the other writers" [36]. In the 1960's Hans Brolin and Pekka John Myrberg studied iterates of real quadratic maps [8], [31].

Then in 1979 Benoît Mandelbrot discovered the set that is now named after him by considering iterations of $z^2 + c$. The book *Fractals and Chaos: The Mandelbrot set and Beyond* contains a wonderful brief history of the mathematics at the time and an outline of the unveiling of such an extraordinary set [27]. In addition, Heinz-Otto Peitgen and Peter H. Richter published a book, *The Beauty of Fractals*, in 1986 which included many beautiful images produced through iteration of functions [33]. These computer images helped rejuvenate the field of Complex Dynamics.

Even prior to Julia and Fatou studying iteration of rational functions, Arthur Cayley and Ernst Schröder were considering, in the 1870's, extending Newton's method to polynomials in the complex plane, instead of the real line [9], [10], [37], [38]. Their goal was to study the basin of attraction of the fixed points (see §2.6 for definition and more details of Cayley's studies).

Today much has been studied about the iteration of rational functions, for instance see, [4], [15], [28] [30]. Newton's iterative root finding method for polynomials is heavily studied as well, see [6], [14], [19], [24], [32], [34], [41], [42], [43]. In addition to studying Newton's method for polynomials, many different variations of Newton's method, e.g., damped Newton's method, and other iterative root finding methods have been studied concerning polynomials, see [1], [2], [21], [22], [23], [29], [35].

The purpose of this dissertation is to use the known information concerning iteration of rational functions to extend the vast knowledge of the Newton map of polynomials to the

Newton map of rational functions, which has not been covered according to our literature search.

In Chapter 2 we will provide some known information concerning iteration of rational functions as well as the iteration of Newton's method. We will also introduce the Mandelbrot set in this chapter.

Our main new result is found in Chapter 3, which categorizes the Newton maps of all rational functions conjugate to quadratic polynomials. A few lemmas will be introduced here as these are used to prove our result. While these lemmas are just stated here they will be proved in subsequent chapters. Along with the categorization we show how this result connects to the Mandelbrot set and state the Julia set of special cases of these rational functions.

In Chapter 4 we prove that no Newton map of a rational function is conjugate to $z^3$. Here we also state why the methods used in the proof of Theorem 8.1.1, which categorizes all degree 2 Newton maps conjugate to quadratic polynomials, do not produce similarly nice results for degree 3 Newton maps.

Chapters 5 and 6 re-introduce the lemmas initially given in Chapter 3. These lemmas state the degree and the fixed points of the Newton map in terms of the original rational functions. These chapters should give a good initial start to understanding that some straightforward information of the Newton map of polynomials cannot be carried onto the Newton map of rational functions. For instance, $\infty$ is always a fixed point of the Newton map of polynomials; however, this is not always the case for rational functions.

We will conclude with a result that shows why the iteration of a non-degree 1 rational function becomes more complicated. In Chapter 9 we consider any degree 2 rational map $F(z)$, not necessarily of the Newton map form, and write out the exact expression for $F^2(z) = F \circ F(z)$ and $F^3(z) = F \circ F \circ F(z)$, the second and third iterations of $F(z)$.

Before we continue, we give some notation that will be used throughout for convenience:

$\overline{\mathrm{lc}}(f)$ = leading coefficient of the numerator of $f(z)$,

$\underline{\mathrm{lc}}(f)$ = leading coefficient of the denominator of $f(z)$,

$\overline{\deg}(f)$ = degree of the numerator of $f(z)$,

$\underline{\deg}(f)$ = degree of the denominator of $f(z)$,

$z_x = \Re(z)$,

$z_y = \Im(z)$,

$\mathbb{Z}_+$ = the set of positive integers.

# CHAPTER 2
# BACKGROUND

This chapter will introduce some basic known information concerning iteration theory. All theorems stated in this chapter are known and many are commonly found in the literature discussing iteration of rational functions, for example [3], [4], [5], [8], [30] and [34]. In addition, many of the proofs may be found in [4], [8] and [24] and some are included here for completeness.

## 2.1 Iteration

The ***n*th iteration** of a function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ will be denoted by

$$f^n(z) = \underbrace{f \circ f \circ \cdots \circ f}_{n-times}(z)$$

where $f^0(z) = z$.

For a specific **initial value** or **seed value**, $z_0$, the iterates may be written as $z_n = f^n(z_0)$.

The standard notation for derivatives, $f', f'', \ldots, f^{(n)}$, will be used. Notice the difference between the $n$th iteration of $f(z)$, $f^n(z)$, and the $n$th derivative of $f(z)$, $f^{(n)}(z)$.

For convenience, rather than denoting the composition of two functions as $f \circ g$, we will be denoting composition as $fg$. When necessary, the operation of multiplication or composition will be explicitly stated for clarity.

## 2.2 Rational Functions

All **rational functions**, $r(z)$, we will consider are reduced rational functions, in other words functions of the form $\frac{p(z)}{q(z)}$, such that $p(z)$ and $q(z)$ have no common factors.

The **degree** of a rational function, $r(z) = \frac{p(z)}{q(z)}$ is

$$\deg(r) = \max\{\deg(p), \deg(q)\},$$

where $\deg(f) = n$ if $f(z) = \sum_{i=0}^{n} a_i z^i$, and $a_n \neq 0$. The degree of the composition of two rational functions is given by: $\deg(fg) = \deg(f)\deg(g)$. This implies the degree of the $n$th iteration is $\deg(r^n) = [\deg(r)]^n$.

By a nontrivial rational function, we mean $r(z) = \frac{p(z)}{q(z)}$ such that $\deg(q) \geq 1$.

### 2.3 Fixed Points

A **fixed point** of a function $f(z)$ is a point $w$ such that $f(w) = w$. All fixed points fall into one of three main categories: repelling, indifferent and attracting, which is determined by their multiplier. The **multiplier** at a fixed point $w$ is

$$\lambda = \begin{cases} |f'(w)| & \text{if } w \in \mathbb{C}, \\ \\ \lim\limits_{w\to\infty} \left|\dfrac{1}{f'(w)}\right| & \text{if } w = \infty. \end{cases}$$

Let $w \in \mathbb{C}_\infty$ be a fixed point and

1. if $\lambda > 1$ then $w$ is a **repelling fixed point**,
2. if $\lambda = 1$ then $w$ is a **indifferent fixed point**,
3. if $\lambda < 1$ then $w$ is a **attracting fixed point**, and

$3^*$. if $\lambda = 0$ then $w$ is a **super-attracting fixed point**.

Let us look at a simple example: $f(z) = z^2 + \frac{1}{5}$. One can solve $f(z) = z$ and find the three fixed points are $\frac{1}{2} + \frac{\sqrt{5}}{10}$, $\frac{1}{2} - \frac{\sqrt{5}}{10}$ and $\infty$. Now, we compute the derivative $f'(z) = 2z$ to find the multiplier $\lambda$. $\left|f'\left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right)\right| = \left|1 + \frac{\sqrt{5}}{5}\right| > 1$, so $\frac{1}{2} + \frac{\sqrt{5}}{10}$ is a repelling fixed point. $\left|f'\left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right)\right| = \left|1 - \frac{\sqrt{5}}{5}\right| < 1$, so $\frac{1}{2} - \frac{\sqrt{5}}{10}$ is an attracting fixed point. And $\left|\frac{1}{f'(\infty)}\right| = 0$, so $\infty$ is a super-attracting fixed point. In fact $\infty$ is a super-attracting fixed point for all polynomials, $p(z)$, with $\deg(p) \geq 2$, since $|p'(\infty)| = \infty$.

The following well-known proposition states that if the iterates of $f$ converge to a point, then this point must be a fixed point of $f$.

**Proposition 2.3.1.** *Let $f$ be a continuous function. If $f^n(z_0) \to w$, as $n \to \infty$, for some $z_0 \in \mathbb{C}_\infty$ then $w$ is a fixed point of $f$.*

*Proof.* Suppose that $f^n(z_0) \to w$, for $z_0 \in \mathbb{C}_\infty$, as $n \to \infty$. Let $z_n = f^n(z_0)$. Therefore, since $f$ is continuous,

$$w = \lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} f(z_n) = f(\lim_{n\to\infty} z_n) = f(w).$$

□

## 2.4 Periodic Points and Cycles

A **periodic point** $w$ is a point such that $f^n(w) = w$ for some integer $n \geq 1$, and $f^j(w) \neq w$, for all $j \in \{1, 2, \ldots, n-1\}$. The set of distinct points $\{w, f(w), f^2(w), \ldots, f^{n-1}(w)\}$ is called the **cycle** of $w$, and $n$ is the **period** of $w$.

By defining the **multiplier at the periodic point** $w$ as

$$\lambda = \begin{cases} |(f^n)'(w)| & \text{if } w \in \mathbb{C}, \\ \\ \lim\limits_{w \to \infty} \left| \dfrac{1}{(f^n)'(w)} \right| & \text{if } w = \infty, \end{cases}$$

we may classify a periodic point as we did in §2.3, as repelling, indifferent or attracting.

Notice, if $w$ is a periodic point with period $n = 1$ then $w$ is actually a fixed point.

## 2.5 Conjugation

A **Möbius transformation** is defined as a degree one rational function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of the form:

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Notice a Möbius transformation is injective, and the inverse, $M^{-1}(z) = \dfrac{dz - b}{-cz + a}$ is also a Möbius transformation.

Two functions $f$ and $g$ are **conjugate**, denoted $f \sim g$, if there exists a Möbius transformation, $M$, such that

$$f = MgM^{-1}.$$

If the specific Möbius transformation should be documented the notation, $f \overset{\overrightarrow{M}}{\sim} g$ or $f \overset{\overleftarrow{M}}{\sim} g$ will be used, i.e., $f \overset{\overrightarrow{M}}{\sim} g$ indicates $f = MgM^{-1}$, while $f \overset{\overleftarrow{M}}{\sim} g$ indicates $MfM^{-1} = g$, or equivalently, $f = M^{-1}gM$.

Conjugation is frequently used with iterations because of the following known properties.

**Proposition 2.5.1.** *(Conjugation preserves degree.) Let $f, g$ be two rational functions. If $f \sim g$ then* $\deg(f) = \deg(g)$.

*Proof.* Recall that the degree of a composition of functions is the product of the degrees, also if $M$ is a Möbius transformation, then so is $M^{-1}$. Hence it follows that

$\deg(M^{-1}) = 1 = \deg(M)$. Therefore,

$$\deg(f) = \deg(MgM^{-1}) = \deg(M)\ \deg(g)\ \deg(M^{-1}) = \deg(g).$$

□

**Proposition 2.5.2.** *If* $f \overset{\overrightarrow{M}}{\sim} g$ *then* $f^n = Mg^nM^{-1}$.

*Proof.* A quick computation gives

$$f^n = \underbrace{MgM^{-1} \circ MgM^{-1} \circ \cdots \circ MgM^{-1}}_{n-\text{times}} = Mg^nM^{-1}.$$

□

Proposition 2.5.2 allows us to conjugate $g$ to $f$ and, instead of studying the iteration of $g$, we may study the iteration of $f$. In practice (see Chapter 3) this will be used when $f$ is a "simpler function" (for instance, $f$ is a polynomial while $g$ is a rational function) or when more properties are known about the iterates of $f$.

**Proposition 2.5.3.** *Let* $f \overset{\overrightarrow{M}}{\sim} g$*, then* $f(z)$ *fixes* $M(\hat{z})$ *if and only if* $g(z)$ *fixes* $\hat{z}$*.*

*Proof.* Let $f \overset{\overrightarrow{M}}{\sim} g$, so that $f(z) = MgM^{-1}(z)$. By evaluating at $M(\hat{z})$ we have

$$f(M(\hat{z})) = MgM^{-1}(M(\hat{z})) = M(g(\hat{z})). \tag{2.1}$$

Suppose first that $f(z)$ fixes $M(\hat{z})$, i.e., $f(M(\hat{z})) = M(\hat{z})$, from which (2.1) implies $M(\hat{z}) = M(g(\hat{z}))$. Since $M(z)$ is a Möbius transformation, it is injective. So it follows from injectivity that $\hat{z} = g(\hat{z})$.

Conversely, if $g(\hat{z}) = \hat{z}$ then (2.1) becomes $f(M(\hat{z})) = M(\hat{z})$, so $f(z)$ fixes $M(\hat{z})$.

□

**Proposition 2.5.4.** *A non-constant rational map* $R(z)$ *is conjugate to a polynomial if and only if there exists* $w \in \mathbb{C}_\infty$ *such that* $R^{-1}(w) = \{w\}$*, the preimage of* $w$ *is a singleton.*

*Proof.* Let $R \overset{\overrightarrow{M}}{\sim} p$, where $R$ is a rational function, $p$ is a polynomial and $M$ is a Möbius transformation. Consider the following equalities:

$$\begin{aligned} R(z) &= MpM^{-1}(z) \\ R\left(Mp^{-1}(z)\right) &= MpM^{-1}\left(Mp^{-1}(z)\right) \\ RMp^{-1}(z) &= M(z) \\ R^{-1}\left(RMp^{-1}(z)\right) &= R^{-1}\left(M(z)\right) \\ Mp^{-1}(z) &= R^{-1}M(z) \end{aligned}$$

Note $p(z)$ is a polynomial if and only if the only pole of $p(z)$ occurs at $\infty$. In other words, the pull-back of $\infty$ only equals $\infty$: $p^{-1}(\infty) = \{\infty\}$.

Thus, since $Mp^{-1}(z) = R^{-1}M(z)$, evaluating at $\infty$ we obtain: $M(\infty) = R^{-1}M(\infty)$. $M(z)$ is a Möbius transformation and is injective; hence there exists exactly one point, say $w \in \mathbb{C}_\infty$, such that $M(\infty) = \{w\}$. Therefore, $\{w\} = R^{-1}\{w\}$, in other words $R^{-1}(w) = \{w\}$.

□

The iteration of a polynomial of degree $d$ involves fewer operations than the iteration of a nontrivial rational function of degree $d$. Because of this, if we know, by Proposition 2.5.4, that a rational function is conjugate to a polynomial, the study of the iteration is simplified. For instance, Proposition 2.5.4 will be used frequently in the proof of Theorem 3.1.4.

Furthermore, if there exists a $w$, for rational function $R(z)$, such that $R^{-1}(w) = \{w\}$, then by Proposition 2.5.4 we know $R \sim g$, for some polynomial $g$ and Proposition 2.5.1 ensures that $\deg(g) = \deg(R)$. Therefore, in practice, our 'search' for a simple function $g$ is much more directed. For example, if we are studying the iteration of a degree five rational function, $R(z)$, and there exists a $w$ such that $R^{-1}(w) = \{w\}$, then we know to focus our attention on polynomials. In addition, we know to focus on a fifth degree polynomial, since the degree of the conjugate functions must be the same.

We are not prepared to state our Corollary to Proposition 2.5.4 at this point, since we have not yet introduced the Newton map and its fixed points. For this reason, Corollary 3.1.3 can be found in Chapter 3 and will be proved in Chapter 6.

## 2.6 Newton's Method

Newton's method, also known as the Newton-Fourier method, is a common iterative process to find the roots of a function, say $r(z)$. The iteration defined by **Newton's**

**method** has the form:

$$z_{n+1} = z_n - \frac{r(z_n)}{r'(z_n)} \qquad \text{for } n = 0, 1, 2, \ldots.$$

The right hand side of this expression can be written as a function $R(z) = z - \dfrac{r(z)}{r'(z)}$, commonly known as the **Newton map**.

The **basin of attraction** of the fixed point $w$ is the set of points that converge to the fixed point under iteration, $\{z \,|\, R^n(z) \to w \text{ as } n \to \infty\}$. For our purposes, rather than using the Newton map to determine the roots of a function, instead we will assume the roots are known and study the basins of attraction of those roots.

In 1879, Cayley applied Newton's method to $z^2 - n^2$, asked and found "under what conditions do we thus approximate to one determinate root" [9]. In the same year, after Cayley found the basin of attraction of the roots $\pm n$, he investigated cubic polynomials and mentioned "the cubic equation appears to present considerable difficulty" [10].

It will be shown in Chapter 3 and then in further detail in Chapter 7 that the Newton map of $z^2 - n^2$, or any quadratic for that matter, is conjugate to $z^2$. Proposition 7.2.1 says the iterates of the Newton map of $z^2 - n^2$ behave similarly to the iterates of $z^2$. This similar behavior is fortunate and one will see why after §2.9 in which the nice behavior of the iteration of $z^2$ will be investigated.

Prior to stating some important theorems about the Newton map we will investigate two examples. The first example is the Newton map of a quadratic polynomial and the second example is the Newton map of a cubic polynomial.

Figure 2.1 shows the basins of attraction of the Newton map's two fixed points, which are the roots of $z^2 - 4$. The blue colored region, the right half plane, is the basin of attraction of root 2, and the green colored region, the left half plane, is the basin of attraction of root $-2$. Notice the initial values closer to $\pm 2$ are in the basin of attraction of $\pm 2$. For example, all points, $z_0$, in the right half plane satisfy $|z_0 - 2| < |z_0 - (-2)|$, and all points in the right-half plane converge to the root 2.

Figure 2.2 shows the basins of attraction of the Newton map of $z^3 - 1$. The teal colored region is the basin of attraction of root 1, the purple colored region is the basin of attraction of root $\frac{-1+i\sqrt{3}}{2}$, and the green colored region is the basin of attraction of root $\frac{-1-i\sqrt{3}}{2}$. It is evident why Cayley had "considerable difficulty" investigating this iteration since the basins of attraction are closely interlaced and the idea of 'closeness' of the initial value, with respect to the roots, as in the Newton map of $z^2 - 4$, does not apply to the

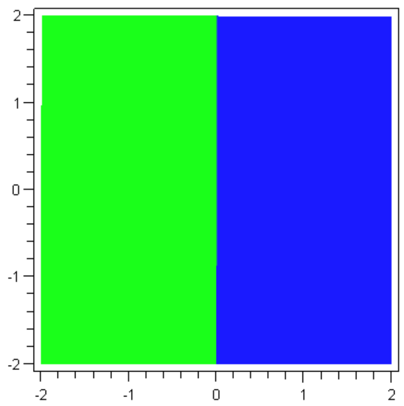


Figure 2.1: Basins of Attraction of the Newton Map of $z^2 - 4$.

basins of attraction of the Newton map of $z^3 - 1$. For instance, if $z_0$ is closer to 1 it does not necessarily mean that $z_n \to 1$, as can be seen in Figure 2.2.

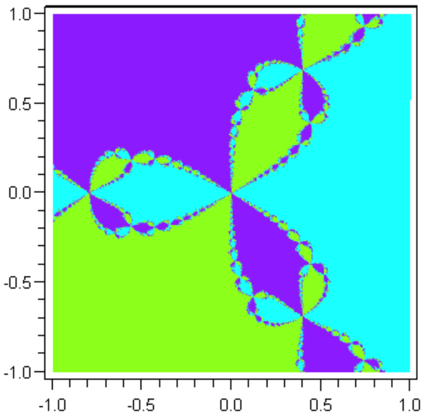


Figure 2.2: Basins of Attraction of the Newton Map of $z^3 - 1$.

The proofs of the following two results follow directly.

**Proposition 2.6.1.** *The Newton map of* $r(z)$ *is equal to the Newton map of* $s(z) = c \cdot r(z)$*, for any* $c \in \mathbb{C}\backslash\{0\}$.

**Corollary 2.6.2.** *The Newton map of* $r(z) = \dfrac{\sum_{i=0}^{n} a_i z^i}{\sum_{j=0}^{m} b_j z^j}$ *is equal to the Newton map of* $s(z) = \dfrac{\sum_{i=0}^{n} \frac{a_i}{a_n} z^i}{\sum_{j=0}^{m} \frac{b_j}{b_m} z^j}$*, where* $a_n, b_m \neq 0$.

Notice the roots of $r(z)$, are the same as the roots of $s(z)$. For example, by Proposition 2.6.1 and Corollary 2.6.2, when considering the Newton map of $s(z) = (2z - 4)(z + 5)$ we can divide by the leading coefficient and instead consider the Newton map of $r(z) = (z-2)(z+5)$, since the Newton maps are equal. Therefore, in general, it suffices to consider the Newton map of a rational function with monic polynomials in both the numerator and denominator. We will do so throughout, except in §7.1 where the standard notation of a Möbius transformation is used.

## 2.7 The Fatou and Julia Sets

The Fatou and Julia sets of a rational function partition the extended complex plane into two disjoint sets, in which the union is the entire plane. Loosely speaking, for any function $R(z)$, the Fatou set is the set in which for all $n$, $R^n(z)$ behaves in a similar manner for all $z$ 'close' to one another; whereas the Julia set is the set in which for all $n$, $R^n(z)$ behaves erratically for all $z$ 'close' to one another. Before we rigorously define these two sets we must define equicontinuity.

A family of functions $\mathscr{F}$ is **equicontinuous at** $z_0$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \epsilon$, for all $f \in \mathscr{F}$. We say $\mathscr{F}$ is **equicontinuous on a set** $G$ if $\mathscr{F}$ is equicontinuous for all $z \in G$.

Equicontinuity guarantees that the family of functions evaluated at $z_0$ behave in a similar manner for all 'close' $z$. Hence the definition of the Fatou set is clear: the **Fatou set** of $R(z)$, denoted $F(R)$, is the largest open set in $\mathbb{C}_\infty$ such that $\{R^n\}_{n=1}^{\infty}$ is equicontinuous. The **Julia set** of $R(z)$, denoted $J(R)$, is the complement with respect to $\mathbb{C}_\infty$ of the Fatou set.

By definition, since $J(R)$ is the complement of an open set in $\mathbb{C}_\infty$, it is closed, hence compact in $\mathbb{C}_\infty$. While the term Julia set is standard, instead of using the term Fatou set, the terms stable set or set of normality is also used. It is clear why the Fatou set is also called the stable set, since iteration must behave in a stable manner for the composition of functions to be equicontinuous. The term "set of normality" will also be clear after the following information is presented.

Traditionally, (for example see [3], [5], [8], [15] and [30]) the Fatou set is defined as the largest domain in $\mathbb{C}_\infty$ such that $\{R^n\}_{n=1}^{\infty}$ is a normal family. Both definitions are equivalent, and this can be seen through the Arzelá-Ascoli Theorem. Recall, **domain** is a connected open subset of $\mathbb{C}_\infty$.

First recall $(f_n)$ converges **locally uniformly** to $f$ in $D$ if for all $z \in D$ there exists a neighborhood on which $(f_n)$ converges uniformly to $f$. In addition, a family of functions, $\mathscr{F}$, from $D$ to $G$ is said to be **normal** in $D$ if every infinite sequence of functions from $\mathscr{F}$ contains a subsequence which converges locally uniformly on $D$.

**Theorem 2.7.1.** *(Arzelá-Ascoli Theorem.) Let D be a subdomain on the complex sphere and let* $\mathscr{F}$ *be a family of continuous maps of D into the sphere. Then* $\mathscr{F}$ *is equicontinuous in D if and only if it is a normal family in D.*

This theorem is proved in many texts, for example see [12] (page 148). Theorem 2.7.1 ensures that our definition of the Fatou set, being the largest domain in $\mathbb{C}_\infty$ such that $\{R^n\}_{n=1}^\infty$ is equicontinuous, is equivalent to the traditional definition of the Fatou set being the largest domain in $\mathbb{C}_\infty$ such that $\{R^n\}_{n=1}^\infty$ is normal.

Before well-established properties of the Fatou and Julia sets are presented the known concept of invariant sets will be addressed, along with a proposition. Let $f : X \to X$ then the set $E \subset X$ is:

1. **forward invariant** if $f(E) = E$,
2. **backward invariant** if $f^{-1}(E) = E$, and
3. **completely invariant** if $f(E) = E = f^{-1}(E)$.

These definitions lead us to the following proposition whose proof is a straightforward application of the definitions.

**Proposition 2.7.2.** *If f is surjective then E is backward invariant if and only if E is completely invariant.*

Note, all rational functions, $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$, are surjective. For this reason we may use backward invariant and completely invariant interchangeably.

The proofs of the remaining standard theorems in this section can be found in many texts, for example see [4], [8] or [15].

**Theorem 2.7.3.** *The Fatou set and the Julia set are completely invariant under any rational map.*

**Theorem 2.7.4.** *For a rational function, R, either* $J(R) = \mathbb{C}_\infty$ *or* $J(R)$ *has empty interior. In fact, if P is a polynomial then* $J(P)$ *has empty interior.*

The next two theorems state that all repelling fixed points lie in the Julia set, while all attracting fixed points lie in the Fatou set.

**Theorem 2.7.5.** *For any rational map, R, every repelling fixed point of R lies in* $J(R)$*.*

**Theorem 2.7.6.** *Let* $w$ *be a (super)attracting fixed point of* $R$*. Then* $w$ *lies in a component, say* $F_j$*, of the Fatou set* $F(R)$*, and as* $n \to \infty$*,* $R^n \to w$ *locally uniformly on* $F_j$*.*

The final two theorems in this section show that equicontinuity is transferred (preserved) as one would like under conjugation (iteration).

**Theorem 2.7.7.** *Let* $R$ *and* $S$ *be non-constant rational maps such that* $S \overset{\overrightarrow{M}}{\sim} R$*. Then* $F(S) = M(F(R))$ *and* $J(S) = M(J(R))$*.*

**Theorem 2.7.8.** *For any non-constant rational map* $R$*, and any positive integer* $n$*,* $F(R^n) = F(R)$ *and* $J(R^n) = J(R)$*.*

## 2.8 Iteration of Möbius Transformations

Recall Möbius transformations have the form

$$M(z) = \frac{az+b}{cz+d}, \text{ where } a,b,c,d \in \mathbb{C} \text{ and } ad-bc \neq 0.$$

Notice, $M(z)$ may be written as a matrix of the form: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{C})$, where the coefficients are placed in the obvious manner.

Let

$$M(z) = \frac{az+b}{cz+d} \qquad \text{and} \qquad N(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$$

be two Möbius transformations. Consider the composition:

$$MN(z) = \frac{a\dfrac{\alpha z+\beta}{\gamma z+\delta}+b}{c\dfrac{\alpha z+\beta}{\gamma z+\delta}+d} = \frac{a(\alpha z+\beta)+b(\gamma z+\delta)}{c(\alpha z+\beta)+d(\gamma z+\delta)} = \frac{z(a\alpha+b\gamma)+(a\beta+b\delta)}{z(c\alpha+d\gamma)+(c\beta+d\delta)}. \tag{2.2}$$

Now consider the multiplication of matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a\alpha+b\gamma & a\beta+b\delta \\ c\alpha+d\gamma & c\beta+d\delta \end{pmatrix}. \tag{2.3}$$

Here we see function composition can be represented by matrix multiplication. By placing the coefficients of $MN$, found in (2.2), we have the same form as (2.3). By induction one can prove that the composition of Möbius transformations is given by matrix multiplication, and because of this the coefficients of the $n$th iterate, $M^n(z)$, can be taken from the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^n$.

A well-known result about Möbius transformations is found in the following proposition.

**Proposition 2.8.1.** *A Möbius transformation $M(z)$ has either two distinct fixed points or a single (repeated) fixed point.*

The proof can be found in many texts, and can also be seen by solving $M(z) = z$ for $z$.

### 2.9 Iteration of the Function $z^2$

We would now like to investigate the iteration of $f(z) = z^2$, but before discussing the details let us examine Figure 2.3. The blue open unit disk in Figure 2.3 is the basin of attraction of the fixed point 0, and the red region is the basin of attraction of the fixed point $\infty$. Note, the union of the two basins of attraction is the Fatou set, and the boundary of the two basins, the unit circle, is the Julia set.

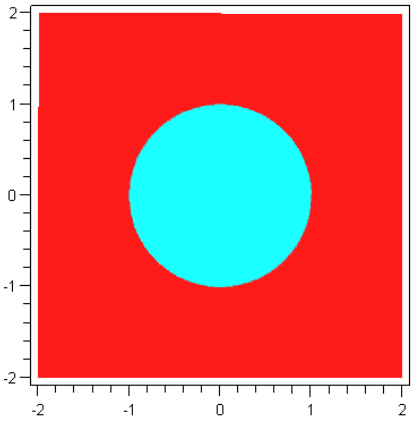


Figure 2.3: Basins of Attraction of the Iteration of the Function $z^2$.

Now let us investigate these claims. Let $f(z) = z^2$, then clearly $f^n(z) = z^{2^n}$. Notice the two attracting fixed points of $f(z)$ are 0 and $\infty$. Using basic complex arithmetic one can see if $|z| < 1$ then $f^n(z) \to 0$ as $n \to \infty$, and if $|z| > 1$ then $f^n(z) \to \infty$ as $n \to \infty$. The iterates in both of these domains: $D_1 = \left\{z \,\middle|\, |z| < 1\right\}$ and $D_2 = \left\{z \,\middle|\, |z| > 1\right\}$, behave in a similar manner, such that any seed value will eventually converge to 0 or $\infty$, depending on the initial location. Notice $D_1$ is the basin of attraction of 0, while $D_2$ is the basin of attraction of $\infty$ since all points in these domains converge to their corresponding fixed points. In addition, it can be shown the union of these two sets is the Fatou set of $f(z) = z^2$. Proving $F(z^2) = D_1 \cup D_2$ is accomplished by showing that for a given $\epsilon$ all points in each domain within some $\delta$-distance of one another, under iteration of $z^2$, end up $\epsilon$-distance of one another.

Now consider $S^1 = \{z \,|\, z = e^{i\theta},\ \text{where } \theta \in [0, 2\pi)\}$. We now will explain the different behaviors of $\{f^n(z)\}_{n=1}^{\infty}$ on $S^1$ in which the iterates do not behave in a stable manner as they did in $D_1$ and $D_2$.

The repelling fixed point of $f(z)$ is 1, which means $|f'(1)| > 1$, or for $z_0$ 'close' to 1, $|f(z_0) - f(1)| > |z_0 - 1|$. Therefore, for some values of $z_0 \in S^1$, $f(z_0)$ is further away from 1 than $z_0$ is from 1. However, $f(-1) = 1$, so there clearly are points in $S^1$ which converge to the fixed point 1.

The action of $f$ on $S^1$ is easier to observe when writing $z$ in polar coordinates. Let $z = e^{i\theta}$; hence, $f(e^{i\theta}) = e^{2i\theta}$ and $f^n(e^{i\theta}) = e^{2^n i\theta}$. Consider two initial values, $z_0 = e^{i\alpha}$ and $\widetilde{z_0} = e^{i\beta}$ such that they are arbitrarily close with respect to arc length, in other words, $|\alpha - \beta|$ is arbitrarily small. Notice the parameterized arc length between $f(z_0)$ and $f(\widetilde{z_0})$ is given by $|f(z_0) - f(\widetilde{z_0})| = |2\alpha - 2\beta|$; therefore, the parameterized distance between $z_0$ and $\widetilde{z_0}$ is doubled by only applying $f(z)$ once. Continuing this iteration process it can be seen, no matter how close $z_0$ and $\widetilde{z_0}$ are initially, they will eventually end up 'far' away; to be exact, $2^n$ times further apart, with respect to the parameterized arc length. Hence, for large enough $n$, any arbitrary small arc on $S^1$ will eventually cover the entire unit circle.

Suppose for some initial value $z_0 = e^{i\theta}$ and for some $n \geq 1$, $f^n(z_0) = 1$. This means for some $n, k \in \mathbb{Z}_+$, $f^n(e^{i\theta}) = e^{2\pi ik}$. In other words, $e^{2^n i\theta} = e^{2\pi ik}$. Thus $2^n i\theta = 2\pi i(k + \ell)$, for some $\ell$, which implies $\theta = \dfrac{\pi(k + \ell)}{2^{n-1}}$. Hence, $z_{m+1} = 1$ if and only if, for $h = k + \ell$, that $z_0 = e^{i\frac{\pi h}{2^m}}$ for some $h, m \in \mathbb{Z}_+$.

Let the set of the points which converge to 1 be denoted as

$$P = \left\{ z \,\middle|\, z = e^{i\frac{\pi k}{2^m}} \text{ for } k, m \in \mathbb{Z}_+ \right\}.$$

It is true that $P$ and $S^1 \backslash P$ are dense in $S^1$, while the details of the proof are technical it can be briefly outlined: Both the diatic and non-diatic rationals are dense in $\mathbb{R}$, and since there is a surjective map from $[0, 1]$ onto $S^1$, it follows that $P$ and $S^1 \backslash P$ are dense in $S^1$. Therefore, an arc of any length in $S^1$ will have infinitely many points in $P$ and in $S^1 \backslash P$ as well. Hence, any arc will have infinitely many points which converge to 1, and infinitely many points which do not converge to 1. This demonstrates a portion of the chaotic behavior of $S^1$, under the iteration of $z^2$.

It can also be shown that $S^1 = J(z^2)$, by showing $\{f^n\}_{n \geq 0}$ is not equicontinuous on the unit circle. For any $\epsilon$, take a small $\delta-$ball around $w \in S^1$ and given large enough $n$, $|f^n(z) - f^n(w)| > \epsilon$. Within the $\delta-$ball around $w \in S^1$ there exist some $z$ such that $z \in D_1$, and, as we mentioned previously, $f^n(z) \to 0$ as $n \to \infty$. In addition, there also exist other $z$ within the same $\delta-$ball such that $z \in D_2$ and $f^n(z) \to \infty$ as $n \to \infty$. Therefore, given large enough $n$ we know $|f^n(z) - f^n(w)| > \epsilon$. In other words, $\{f^n\}_{n \geq 0}$ is not equicontinuous on the unit circle, and $S^1 = J(z^2)$.

Note $f(z) = z^2$ is not unique in its properties of having $D_1$ as the basin of attraction of 0 and $D_2$ as the basin of attraction of $\infty$ (i.e. $F(z^2) = D_1 \cup D_2$), as well as having $S^1$ exhibit chaotic behavior (i.e. $J(z^2) = S^1$). In fact, for $d \in \mathbb{Z}_+$, $f(z) = z^d$ has these same properties. While the points which converge to 1 will vary, there are still only a countable number of points that lie on $S^1$ which converge to 1. All can be proved using the same methods as were used above.

## 2.10 Iteration of Quadratic Polynomials

A well known and very useful conjugation of quadratic polynomials, for our study, can be seen in the following theorem, also see [6].

**Theorem 2.10.1.** *Every complex quadratic polynomial of the form* $p(z) = az^2 + 2bz + d$ *is conjugate to:*

$$r_c(z) = z^2 + c$$

*such that* $p \overset{\overleftarrow{M}}{\sim} r_c$ *where,* $M(z) = az + b$ *and* $c = ad + b - b^2$.

*Proof.* Let $M(z) = \alpha z + \beta$, where $\alpha \neq 0$, and $r_c(z) = z^2 + c$, then

$$M^{-1} r_c M(z) = \alpha z^2 + 2\beta z + \frac{c + \beta^2 - \beta}{\alpha}.$$

Solving

$$\alpha z^2 + 2\beta z + \frac{c + \beta^2 - \beta}{\alpha} = p(z) = az^2 + 2bz + d$$

for the unknowns $\alpha, \beta$ and $c$ we obtain $\alpha = a, \beta = b$ and $c = ad + b - b^2$. Hence, we have our conjugated polynomial, $z^2 + (ad + b - b^2)$ , as well as our Möbius transformation, $M(z) = az + b$.

□

We note here two other equivalent forms for a quadratic polynomial. As previously, let

$c = ad + b - b^2$ and $p(z) = az^2 + 2bz + d$. Then $p \overset{\overleftarrow{N}}{\sim} (z^2 + \lambda z)$ where

$$N(z) = az + b + \frac{-1 + \sqrt{1 - 4c}}{2} \quad \text{and} \quad \lambda = 1 - \sqrt{1 - 4c}.$$

As well as, $p \overset{\overleftarrow{g}}{\sim} (\mu z(1 - z))$, also called the logistic map, where

$$g(z) = -\frac{a}{1 + \sqrt{1 - 4c}} z - \frac{2b - 1 - \sqrt{1 - 4c}}{2 + 2\sqrt{1 - 4c}} \quad \text{and} \quad \mu = \frac{2(1 - \sqrt{1 - 4c} - 2c)}{1 + \sqrt{1 - 4c}}.$$

A nice result concerning the iteration of $z^2 + c$ is found in the following proposition.

**Proposition 2.10.2.** *The function $r_c(z) = z^2 + c$ has at most one finite attracting fixed point.*

*Proof.* By way of contradiction, suppose $\alpha$ and $\beta$ are two attracting fixed points of $r_c(z)$. This means that $r_c(w) = w$ and $|r_c'(w)| < 1$ for $w = \alpha, \beta$. Suppose $r_c(z) = z^2 + c = z$, solving for $z$ we obtain $z = \frac{1 \pm \sqrt{1 - 4c}}{2}$. Notice that $\frac{1 + \sqrt{1 - 4c}}{2} + \frac{1 - \sqrt{1 - 4c}}{2} = 1$, and since $\alpha, \beta$ are these fixed points this implies $\alpha + \beta = 1$. Now consider

$$r_c'(\alpha) + r_c'(\beta) = 2\alpha + 2\beta = 2(\alpha + \beta) = 2.$$

This is a contradiction, since $2 = |r_c'(\alpha) + r_c'(\beta)| \leq |r_c'(\alpha)| + |r_c'(\beta)| < 1 + 1 = 2$. Therefore, there exists at most one finite attracting fixed point.

□

Throughout we will focus on the conjugated form $r_c(z) = z^2 + c$, because of what is known about the constant $c$ and the Mandelbrot set as discussed next.

## 2.11 The Mandelbrot Set

All information presented in this subsection is well established, and the information presented here, including some proofs are added for completeness.

The **Mandelbrot set** is in the $c$-plane and is defined as:

$$\mathscr{M} = \left\{c \,\middle|\, r_c^n(0) \nrightarrow \infty\right\}, \text{ where } r_c(z) = z^2 + c.$$

An equivalent definition is the set of $c$ such that the Julia set of $z^2 + c$ is connected [16]. The Mandelbrot set was defined by Benoît Mandelbrot as the set of $c$ such that the sequence $\{c, c^2 + c, (c^2 + c)^2 + c, \ldots\}$ is bounded [27].

Figure 2.4 shows an approximation of the Mandelbrot set. Rough images of this set were first seen by Mandelbrot in 1979, see [27] for one of the first images produced of this set. We will discuss later how this figure was constructed.

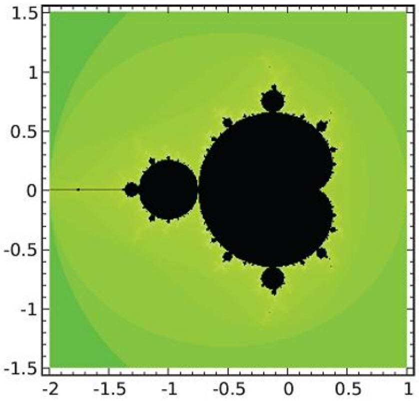


Figure 2.4: The Mandelbrot Set.

A very useful fact for constructing the $\mathscr{M}$ set is found in the following theorem, and the proof may be found in [7].

**Theorem 2.11.1.** *Let* $z_n = z_{n-1}^2 + c$. *If* $|z_i| > 2$ *for some* $i$ *then* $z_n \to \infty$ *as* $n \to \infty$. *In other words, if* $|r_c^n(0)| \leq 2$ *for all* $n$ *then* $c \in \mathscr{M}$.

The **filled Julia set** is the set of all $z_0$ such that $z_n \nrightarrow \infty$ as $n \to \infty$, and is usually denoted as $K_c = \left\{ z_0 \,\middle|\, r_c^n(z_0) \nrightarrow \infty \right\}$. All finite attracting and repelling fixed points of $r_c(z)$ lie in $K_c$. It should be clear why the finite attracting fixed points lie in $K_c$. As for the repelling fixed points, recall by Theorem 2.7.5 that the repelling fixed points lie in the Julia Set. Also $J(r_c) \subset K_c$, which can be seen by observing that the set $\mathbb{C}_\infty \backslash K_c$ is the set of initial points which converge to $\infty$, i.e. no chaotic behavior occurs, all points converge to $\infty$.

Comparing $K_c$ and $\mathscr{M}$ we see that $0 \in K_c$ if and only if $c \in \mathscr{M}$. Much can be said about $r_c^n(z)$ by investigating the initial value of $z_0 = 0$. For instance, if 0 is such that $r_c^n(0) \nrightarrow \infty$, i.e. $0 \in K_c$, then the Julia set of $z^2 + c$ is connected [15]. However, one must be careful, $K_c$ is different than $\mathscr{M}$. Notice that $K_c$ contains all the initial value points $z_0$ that do not converge to $\infty$ for a *fixed* $c$, and $\mathscr{M}$ contains *all* the $c$ such that the fixed initial value 0 does not converge to $\infty$. $\mathscr{M}$ is a fixed set, while $K_c$ changes depending upon the $c$.

Notice, $\mathscr{M}$ appears to be symmetric with respect to the real axis in Figure 2.4. This observation is well known, and we will use the following lemma to prove this symmetry.

**Lemma 2.11.2.** $\overline{r_{\bar{c}}^k(0)} = r_c^k(0)$.

The proof follows from an easy induction argument.

**Proposition 2.11.3.** *If $c \in \mathscr{M}$ then $\bar{c} \in \mathscr{M}$.*

*Proof.* Let $c \in \mathscr{M}$. By Theorem 2.11.1 we have that for all $n$, $|r_c^n(0)| \leq 2$. Using Lemma 2.11.2, and the property that for all $z \in \mathbb{C}$, $|z| = |\bar{z}|$, we have the following:

$$|r_{\bar{c}}^n(0)| = |\overline{r_{\bar{c}}^n(0)}| = |r_c^n(0)| \leq 2.$$

Therefore, if $c \in \mathscr{M}$ then $\bar{c} \in \mathscr{M}$.

□

If one looks closely at Figure 2.4 it can be seen that part of the negative real axis is included and it appears that $\mathscr{M} \cap \mathbb{R} = [-2, 1/4]$. We will use the following lemma to prove this observation is true.

**Lemma 2.11.4.** *Let $r_c(z) = z^2 + c$.*

*(i) If $c \in [-2, 0)$, then $r_c([-\beta, \beta]) \subseteq [-\beta, \beta]$, where $\beta = \dfrac{1 + \sqrt{1-4c}}{2}$.*

*(ii) If $c \in \left[0, \frac{1}{4}\right]$, then $r_c([0, \alpha]) \subseteq [0, \alpha]$ into itself, where $\alpha = \dfrac{1 - \sqrt{1-4c}}{2}$.*

*(iii) If $c > \frac{1}{4}$, then $r_c^n(0) \geq n\left(c - \frac{1}{4}\right)$.*

*Proof.* (*i*) Let $c \in [-2, 0)$. Since $r_c(z) = z^2 + c$ is a continuous function on the closed interval $[-\beta, \beta]$, it attains its maximum and minimum values. The minimum is $c$, and the maximum occurs at the end points $\pm\beta$, and is $\beta^2 + c$. We must verify $c, \beta^2 + c \in [-\beta, \beta]$ to show $r_c([-\beta, \beta]) \subseteq [-\beta, \beta]$.

When $c < 0$ it should be clear that $\beta > 0$, so if $c \geq -\beta$, then $c \in [-\beta, \beta]$. We want to verify the following,

$$\begin{aligned} c &\geq -\beta \\ c &\geq -\frac{1 + \sqrt{1-4c}}{2} \\ 0 &\geq 2c + 1 + \sqrt{1-4c}, \end{aligned}$$

for $c \in [-2, 0)$. Let $f(c) = 2c + 1 + \sqrt{1-4c}$, then $f'(c) = 2 - \dfrac{2}{\sqrt{1-4c}}$, and for $-2 \leq c < 0$, we see that $1 < f'(c) \leq 3$. Therefore $2c + 1 + \sqrt{1-4c}$ is increasing for $c \in [-2, 0)$. Hence the minimum occurs at $c = -2$, which is 0. Therefore, $c \geq -\beta$.

Now to show that $\beta^2 + c \in [-\beta, \beta]$ is simply done by verifying $\beta^2 + c = \beta$.

Since $r_c(z)$ is a continuous function on the closed interval and attains its maximum and minimum values $\beta, -\beta$, respectively, $r_c([-\beta, \beta]) \subseteq [-\beta, \beta]$ when $c \in [-2, 0)$.

$(ii)$ Let $c \in \left[0, \frac{1}{4}\right]$. Again, since $r_c(z)$ is a continuous function on the closed interval $[0, \alpha]$ it obtains its maximum and minimum values. The minimum is $c$, which occurs at 0, and the maximum occurs at $\alpha$, and is $\alpha^2 + c$. It is clear that $0 \in \left[0, \frac{1}{4}\right]$, and a quick calculation yields $\alpha^2 + c = \alpha \in \left[0, \frac{1}{4}\right]$ Therefore, $r_c([0, \alpha]) \subseteq [0, \alpha]$.

$(iii)$ By induction, let $n = 0, 1$ then clearly

$$\begin{aligned} r_c^0(0) &= 0 \geq \ 0, \\ r_c^1(0) &= c \geq \ \left(c - \frac{1}{4}\right). \end{aligned}$$

Suppose the result holds for $n = k$, i.e., $r_c^k(0) \geq k\left(c - \frac{1}{4}\right)$.

Consider $n = k + 1$:

$$\begin{aligned} r_c^{k+1}(0) &= r_c(r_c^k(0)) \\ &= \left(r_c^k(0)\right)^2 + c \\ &\geq \left[k\left(c - \frac{1}{4}\right)\right]^2 + c \\ &= k^2\left(c - \frac{1}{4}\right)^2 + c. \end{aligned}$$

Therefore, we know that $r_c^{k+1}(0) \geq k^2\left(c - \frac{1}{4}\right)^2 + c$. If we can show $k^2 + \left(c - \frac{1}{4}\right)^2 + c \geq (k+1)\left(c - \frac{1}{4}\right)$, in other words,

$$k^2\left(c - \frac{1}{4}\right)^2 + c - (k+1)\left(c - \frac{1}{4}\right) \geq 0,$$

then our conclusion holds.

Consider $f(k) = k^2\left(c - \frac{1}{4}\right)^2 - k\left(c - \frac{1}{4}\right) + c - \left(c - \frac{1}{4}\right)$. A simple calculation shows the minimum of $f(k)$ occurs at $k_0 = \dfrac{1}{2\left(c - \frac{1}{4}\right)}$, and we see

$$k^2\left(c - \frac{1}{4}\right)^2 - k\left(c - \frac{1}{4}\right) + c - \left(c - \frac{1}{4}\right) = f(k) \geq f\left(k_0\right) = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} = 0 \geq 0,$$

as desired.

□

A simple application of Lemma 2.11.4 will prove the following proposition.

**Proposition 2.11.5.** $\mathscr{M} \cap \mathbb{R} = \left[-2, \frac{1}{4}\right]$.

The proof is outlined in [4] and the details will be given here as well.

*Proof.* Both Theorem 2.11.1 and Lemma 2.11.4 (*iii*) show for real valued $c \notin \left[-2, \frac{1}{4}\right]$, there exists an $n$ such that $|r_c^n(0)| > 2$. Thus $c \notin \mathscr{M}$.

For $\alpha, \beta$ defined in Lemma 2.11.4 we have the following conclusions. For $c \in [-2, 0)$, $r_c$ maps $[-\beta, \beta]$ into itself and $\max_{c \in [-2,0)} |\beta| = 2$, by Lemma 2.11.4 (*i*). Therefore, since $0 \in [-\beta, \beta]$ we know that $r_c^n(0) \in [-\beta, \beta] \subseteq [-2, 2]$, for all $n$. This implies that $|r_c^n(0)| \leq 2$, for all $n$ and $c \in [-2, 0)$. In addition, for $c \in \left[0, \frac{1}{4}\right]$, Lemma 2.11.4 (*ii*) implies that $r_c$ maps $[0, \alpha]$ into itself, and $\max_{c \in \left[0, \frac{1}{4}\right]} |\alpha| = \frac{1}{2}$. Since $0 \in [0, \alpha]$, we know that $r_c^n(0) \in [0, \alpha] \subseteq \left[0, \frac{1}{2}\right]$, for all $n$. This implies that $|r_c^n(0)| < 2$, for all $n$ and for $c \in \left[-2, \frac{1}{4}\right]$.

□

Now we will mention how Figure 2.4 was constructed. Theorem 2.11.1 was used when plotting the Mandelbrot set in Figure 2.4 and is commonly used to plot the Mandelbrot set. For $c \in \mathbb{C}_\infty$ and $|c| \leq 2$ we fix $N \in \mathbb{N}$ and compute $z_{n+1} = z_n + c$ where $z_0 = 0$ for $n = 0, 1, \ldots, N$. If $|z_n| > 2$ for any $n = 0, 1, \ldots N$, then plot $c$ outside of the Mandelbrot set. Otherwise, if $|z_n| \leq 2$ for all $n = 0, 1, \ldots, N$, then plot $c$ inside the Mandelbrot set. However, note for a fixed $N$, if $|z_n| \leq 2$ for all $n = 0, 1, \ldots, N$ that this does not guarantee that $|z_{N+1}| \leq 2$. Therefore, this method does not provide an exact image of the Mandelbrot set, and depending on the magnitude of $N$ will cause the image of the Mandelbrot set to be larger than it actually is. However, for sufficiently large $N$ one can obtain a close approximation.

Finally, looking closely at Figure 2.4, there is a variation of shade outside of the Mandelbrot set. This shade is determined by how ‘quickly’ the iterations converge to $\infty$, the darker the green the quicker the iterates converge to $\infty$.

We mentioned prior to Lemma 2.11.4 a reason why the initial value of the iteration of $r_c(z)$ is $z_0 = 0$, by comparing $K_c$ and $\mathscr{M}$. The following definition and theorem provide yet another reason for using $z_0 = 0$.

Notice that $r_c'(z) = 2z$, so the **critical point**, the only point $z$ where $r_c$ is not injective in any neighborhood of $z$, is $z = 0$. This definition and observation are important for the following proposition attributed to both Julia and Fatou.

**Proposition 2.11.6.** *Let R be a rational map of degree at least two. Then the immediate basin of each (super-)attracting cycle of R contains a critical point of R.*

*Proof.* See [26].

□

Thus, if $r_c(z)$ has either an attracting fixed point or an attracting cycle then 0 will converge to this point or cycle.

We will now make some observations about the different ‘parts’ of the Mandelbrot set, nothing is proved in detail here, but more information can be found in [4], [16] and [27], for example.

In Figure 2.4 there is a large cardioid and tangent to this are many bulbs. Recall from Proposition 2.10.2 that $r_c(z)$ has at most one finite attracting fixed point. The main cardioid of the Mandelbrot set consists of all $c$ such that $r_c(z)$ has exactly one finite attracting fixed point. To see that the main cardioid consists of all $c$ such that $r_c(z)$ has exactly one finite attracting fixed point, consider the set of all attracting fixed points which can be seen as the set of points such that $|r_c'(z)| = 2|z| < 1$, in other words, all $z$ such that $|z| < \frac{1}{2}$. Thus, the set of all attracting fixed points is the disk $\left\{c \,\middle|\, |c| < \frac{1}{2}\right\}$. We know $z$ is a fixed point if the equation $z^2 + c = z$, or equivalently $c = z - z^2$ holds true. Mapping the disk $\left\{c \,\middle|\, |c| < \frac{1}{2}\right\}$ under $z - z^2$ we obtain the cardioid.

We also mention that the disk $\left\{c \,\middle|\, |c + 1| < \frac{1}{4}\right\}$ is the set of $c$ such that $r_c$ has an attracting 2-cycle. A detailed look at this can be found in [4].

Another important detail that can be seen from Figure 2.4, is that $z = -\frac{3}{4}$ is the point of tangency of the main cardioid and disk $\left\{c \,\middle|\, |c| < \frac{1}{2}\right\}$. This parameter, $c = -\frac{3}{4}$, is called an **unstable parameter**, in other words, there are parameters arbitrarily close to $-\frac{3}{4}$, say $\hat{c}$, such that the behavior of $\{r_{\hat{c}}(z)\}_{n\geq 0}$ is fundamentally different the behavior of $\left\{r_{-\frac{3}{4}}(z)\right\}_{n\geq 0}$, we describe the difference of behavior later in the paragraph. The point $z = -\frac{3}{4}$ is also called a **bifurcation point**, a point on the boundary of two regions where the corresponding dynamics undergo a fundamental change [40]. More details of this may be found in [4], [17], [40]. However, we will mention here that the different dynamics can be seen when $c$ lies in the main cardioid, $r_c(z)$ has one finite attracting fixed point and one repelling fixed point. For $\hat{c}$ in the disk $\left\{c \,\middle|\, |c + 1| < \frac{1}{4}\right\}$, $r_{\hat{c}}(z)$ has an attracting 2−cycle. Finally, at the point $c = -\frac{3}{4}$, we know that $r_{-\frac{3}{4}}(z)$ has a neutral (repeated) fixed point. Hence, for all $c \in \mathscr{M}$ and arbitrarily close to $\tilde{c} = -\frac{3}{4}$ the iterates of $r_c(z)$ can have one of three vastly different outcomes.

While $c = -\frac{3}{4}$ is special in the sense that it indicates the moment of change $c = -\frac{3}{4}$ is not the unique bifurcation point of $\mathscr{M}$. Notice there are many bulbs attached to the main cardioid, as well as smaller ones attached to these. These connections are through points of tangency, and each tangent point is a bifurcation point.

Finally, a few last points, in 1984, Adrien Douady and John Hubbard proved that the Mandelbrot set is connected [18]. This was done by using the Riemann mapping theorem, and mapping the complement of the Mandelbrot set onto the complement of the closed unit disk. It was in fact Douady who proposed the Mandelbrot set be named after Benoît Mandelbrot.

# CHAPTER 3
# CLASSIFICATION OF DEGREE TWO NEWTON MAPS CONJUGATE TO POLYNOMIALS

We will now state a new result that answers the question:

The Newton map of what rational functions are conjugate to a quadratic polynomial?

In addition to answering this question completely we will also explicitly give the Julia set of the Newton maps which are conjugate to $z^2$. The proof of Theorem 8.1.1 relies heavily on the following three results which we will state here and prove later, in Chapters 5 and 6.

Recall, if $r(z) = \frac{p(z)}{q(z)}$ then $\overline{\deg}(r) = \deg(p)$, and $\underline{\deg}(r) = \deg(q)$.

**Lemma 3.1.1.** *Let $r(z)$ be a reduced rational function such that $\overline{\deg}(r) = d$, $\underline{\deg}(r) = e$. Let $n \le d$ and $m \le e$ be the number of distinct roots and poles, respectively. Let $R(z)$ be the Newton map of $r(z)$, then*

$$\deg(R) = \begin{cases} n+m & \textit{if } d \neq e+1 \\ n+m-1 & \textit{otherwise.} \end{cases}$$

The following lemma is stated as one result; however, in Chapter 6 when the result is proved it will be separated into three parts to make the proofs easier to follow.

**Lemma 3.1.2.** *Let $r(z)$ be a reduced rational function, where $\overline{\deg}(r) = d$, $\underline{\deg}(r) = e$, and $R(z)$ is its Newton map. Then the finite (simple-)roots and poles of $r(z)$ are the only finite (super-)attracting and repelling fixed points of the Newton map $R(z)$, respectively. In addition, $\infty$ is a fixed point if and only if $d \neq e+1$. Moreover, in this case, $\infty$ is*

$$\begin{cases} \textit{a super-attracting fixed point} & \textit{for} & d = e, \\ \textit{an attracting fixed point} & \textit{for} & d < e, \\ \textit{a repelling fixed point} & \textit{for} & d > e+1, \\ \textit{an indifferent fixed point} & \textit{never.} & \end{cases}$$

We may use Lemma 3.1.2 to explicitly describe the given $w$ in Proposition 2.5.4 for the specific rational function we are considering, the Newton map. The following corollary does this nicely. It says that the Newton map $R$ is conjugate to a polynomial if and only if there exists a fixed point whose inverse is a singleton.

**Corollary 3.1.3.** *(to Theorem 2.5.4) Let $R(z)$ be the Newton map of $r(z)$, then $R(z)$ is conjugate to a polynomial if and only if at least one of the following occurs:*

1. *there exists a finite root, $r$, of $r(z)$ such that $R^{-1}(r) = \{r\}$,*
2. *there exists a finite pole, $p$, of $r(z)$ such that $R^{-1}(p) = \{p\}$,*
3. *provided $\overline{\deg}(r) \neq \underline{\deg}(r) + 1$, then $R^{-1}(\infty) = \{\infty\}$.*

We will first give an outline of the proof that answers our main question stated above. To answer our question we will consider all reduced complex rational functions, $r(z)$, and show that the four cases stated are the only possibilities such that $R(z) \overset{\overleftarrow{M}}{\sim} z^2 + c$. Note that throughout we are considering $r(z)$ where the roots and the poles may lie in $\mathbb{C}$ and are looking for the range of the multiplicities of the roots and poles.

First, recall Proposition 2.5.1 states that conjugation preserves degree; therefore, if we are interested in finding all Newton maps which are conjugate to degree two polynomials then it follows that the Newton map must be of degree two itself. Notice that Lemma 3.1.1 reduces our proof into two main cases: (1) $n + m = 2$ when $d \neq e + 1$ and (2) $n + m = 3$ when $d = e + 1$.

The final main result we will use in the proof, and have previously stated here, is Corollary 3.1.3 which states that the only possible $w$ which exist satisfying $R^{-1}(w) = \{w\}$, are the roots and poles (and $\infty$, given $\overline{\deg}(r) \neq \underline{\deg}(r) + 1$). Thus, we will suppose there exists a $w$ such that $R^{-1}(w) = \{w\}$ and determine the multiplicities of the roots and poles that satisfy this equation, or verify that none such exist.

**Theorem 3.1.4.** *Let $R(z)$ be the Newton map of $r(z)$. Then $R(z) \overset{\overleftarrow{M}}{\sim} z^2 + c$ if and only if $r(z)$ is of one of the following forms, where $p, r_1, r_2 \in \mathbb{C}$, $r_1 \neq r_2$, $r_i \neq p$ for $i = 1, 2$ and $d \in \mathbb{Z}_+$:*

*(i)* $r(z) = (z - r_1)(z - r_2)^d,$

*(ii)* $r(z) = \dfrac{(z - r_1)^d}{(z - p)^d},$

*(iii)* $r(z) = \dfrac{z - r_1}{(z - p)^d},$

*(iv)* $r(z) = \dfrac{(z - r_1)(z - r_2)^d}{(z - p)^d}.$

*In these cases $c$ and $M(z)$ are given by:*

*(i)* $c = \frac{1}{4} - \frac{1}{4d^2}, \qquad M(z) = \frac{-z(1+d) - r_1 + 2r_2 + r_1 d}{2d(r_1 - z)},$

*(ii)* $c = \frac{1}{4} - \frac{1}{4d^2}, \qquad M(z) = \frac{-1}{d(r_1 - p)}z + \frac{r_1 + p + d(r_1 - p)}{2d(r_1 - p)},$

*(iii)* $c = \frac{1}{4} - \frac{1}{4d^2}, \qquad M(z) = \frac{z(1-d) + (r_1 + dr_1 - 2p)}{2d(r_1 - z)},$

*(iv)* $c = \frac{1}{4} - \frac{1}{4d^2},$

$$M(z) = \frac{z(p - 2r_1 + r_2 + pd - r_2 d) + dr_1 r_2 + r_1 p + r_1 r_2 - 2r_2 p - dr_1 p}{2d(r_1 - z)(r_2 - p)}.$$

*Proof.* Clearly all degree two complex rational functions are of the form:

$$R(z) = \frac{Az^2 + Bz + C}{Dz^2 + Ez + F} \tag{3.1}$$

where $A, B, C, D, E, F \in \mathbb{C}$. Note that for $\deg(R) = 2$ we must have both $A, D \neq 0$. A simple computation provides the inverse image:

$$R^{-1}(z) = \frac{zE - B \pm \sqrt{(B - zE)^2 - 4(A - zD)(C - zF)}}{2(A - zD)}. \tag{3.2}$$

Given $z$ is either a finite root or pole of $r(z)$, or in specific cases $\infty$, we want to find conditions on the multiplicity of the roots and poles for the radicand to vanish, in other words,

$$(B - zE)^2 - 4(A - zD)(C - zF) = 0, \tag{3.3}$$

or to verify no conditions exist for this to occur.

Therefore, for a finite $w$ to satisfy $R^{-1}(w) = \{w\}$ it must satisfy equation (3.3).

If instead $w$ is infinite, we must be more careful. Recall, by Lemma 6.1.4, $R(z)$ has an infinite fixed point if and only if $\overline{\deg}(r) \neq \underline{\deg}(r) + 1$. So our question now becomes what are the conditions such that

$$R^{-1}(\infty) = \{\infty\}.$$

Notice $\infty$ is a fixed point of the degree two Newton map in equation (3.1) only when $D = 0$. Rewriting (3.2) with $D = 0$ we simplify,

$$R^{-1}(z) = \frac{zE - B \pm \sqrt{z^2E^2 + z(4AF - 2BE) + B^2 - 4AC}}{2A}. \tag{3.4}$$

Now for convenience let

$$\begin{aligned} f(z) &= zE - B, & (3.5)\\ g(z) &= \sqrt{z^2E^2 + z(4AF - 2BE) + B^2 - 4AC}. & (3.6) \end{aligned}$$

so that

$$R^{-1}(z) = \frac{f(z) \pm g(z)}{2A}. \tag{3.7}$$

At this point we want to find conditions, if any, that must be held by the variables $A, B, C, E, F$ (recalling $D = 0$), for $R^{-1}(\infty) = \{\infty\}$.

It is clear that,

$$\lim_{z\to\infty} f(z) = \begin{cases} -B < \infty & \text{when } E = 0, \\ \infty & \text{otherwise,} \end{cases}$$

and

$$\lim_{z\to\infty} g(z) = \begin{cases} \sqrt{B^2 - 4AC} < \infty & \text{when } E = 0 \text{ and } F = 0, \\ \infty & \text{otherwise.} \end{cases}$$

Notice, in equation (3.7), $\lim_{z\to\infty} R^{-1}(z)$ has four different sum/difference possibilities:

Case (*i*): $\infty \pm \infty$,

Case (*ii*): $-B \pm \infty$,

Case (*iii*): $\infty \pm \sqrt{B^2 - 4AC}$,

Case (*iv*): $-B \pm \sqrt{B^2 - 4AC}$.

Let us first consider Case (*i*). Notice Case (*i*) implies $R^{-1}(\infty) = \{\infty\}$ if and only if the indeterminate form of "$\infty - \infty$" is infinite itself.

Since $\lim_{z\to\infty} f(z) = \lim_{z\to\infty} g(z) = \infty$, instead of considering equation (3.7), it is sufficient to consider

$$\lim_{z\to\infty}[f(z) - g(z)].$$

We may rewrite the indeterminate form as

$$\lim_{z\to\infty}\left[\frac{\frac{1}{g(z)} - \frac{1}{f(z)}}{\frac{1}{f(z)g(z)}}\right].$$

Using L'Hôpital's rule this limit is equal to

$$\lim_{z\to\infty} \frac{g'(z)[f(z)]^2 - f'(z)[g(z)]^2}{f'(z)g(z) - g'(z)f(z)}. \tag{3.8}$$

Substituting functions (3.5) and (3.6) into (3.8) and simplifying we have

$$\begin{aligned}
(3.8) \quad = \quad & \lim_{z\to\infty} \frac{z^3E^4 + (2AE^2F - 3BE^3)z^2 + (-4ABEF + 3B^2E^2)z + 2AB^2F - B^3E}{2E^3z^2 + (-4BE^2 + 6AEF)z - 4ACE - 2ABF + 2B^2E} \\
- \quad & \frac{E(z^2E^2 + z(4AF - 2BE) + B^2 - 4AC)\sqrt{z^2E^2 + z(4AF - 2BE) + B^2 - 4AC}}{2E^3z^2 + (-4BE^2 + 6AEF)z - 4ACE - 2ABF + 2B^2E} \\
= \quad & \lim_{z\to\infty} \frac{zE^4 + (2AE^2F - 3BE^3) + \frac{-4ABEF+3B^2E^2}{z} + \frac{2AB^2F-B^3E}{z^2}}{2E^3 + \frac{-4BE^2+6AEF}{z} + \frac{-4ACE-2ABF+2B^2E}{z^2}} \qquad (3.9) \\
& - \frac{\frac{E(z^2E^2+z(4AF-2BE)+B^2-4AC)}{z} \cdot \frac{\sqrt{z^2E^2+z(4AF-2BE)+B^2-4AC}}{z}}{2E^3 + \frac{-4BE^2+6AEF}{z} + \frac{-4ACE-2ABF+2B^2E}{z^2}}.
\end{aligned}$$

Note, we may separate (and combine) the limit of sums and products since all limits exist in the chordal metric (see Chapter 6 for the definition of chordal metric). Therefore, evaluating the indeterminate form continues:

$$\begin{aligned}
(3.9) \quad = \quad & \frac{\lim\limits_{z\to\infty}\left(zE^4 + 2AE^2F - 3BE^3\right) + \lim\limits_{z\to\infty}\left(\frac{-4ABEF + 3B^2E^2}{z} + \frac{2AB^2F - B^3E}{z^2}\right)}{\lim\limits_{z\to\infty}\left(2E^3 + \frac{-4BE^2 + 6AEF}{z} + \frac{-4ACE - 2ABF + 2B^2E}{z^2}\right)} \\
- \quad & E \cdot \frac{\left[\lim\limits_{z\to\infty}\left(zE^2 + 4AF - 2BE\right) + \lim\limits_{z\to\infty}\left(\frac{B^2-4AC}{z}\right)\right]\left[\lim\limits_{z\to\infty}\sqrt{E^2 + \frac{4AF-2BE}{z} + \frac{B^2-4AC}{z^2}}\right]}{\lim\limits_{z\to\infty}\left(2E^3 + \frac{-4E^2B + 6AEF}{z} + \frac{-4ACE - 2ABF + 2EB^2}{z^2}\right)} \\
= \quad & \frac{\lim\limits_{z\to\infty}\left(zE^4 + 2AFE^2 - 3BE^3\right)}{2E^3} - E\frac{\left[\lim\limits_{z\to\infty}\left(zE^2 + 4AF - 2BE\right)\right] \cdot [E]}{2E^3} \\
= \quad & \lim_{z\to\infty}\left(\frac{zE^4 + 2AE^2F - 3BE^3 - zE^4 - E^2(4AF - 2BE)}{2E^3}\right) \\
= \quad & \lim_{z\to\infty}\left(\frac{2AE^2F - 3BE^3 - E^2(4AF - 2BE)}{2E^3}\right) = k < \infty.
\end{aligned}$$

for some $k \in \mathbb{C}$.

Therefore, Case (*i*) can be excluded in our search for conditions since $R^{-1}(\infty) = \{k, \infty\}$ where $k \neq \infty$.

Note, Cases (*ii*) and (*iii*) imply $R^{-1}(\infty) = \{\infty\}$, since we are considering rational functions in $\mathbb{C}_\infty$, and $k \pm \infty = \infty$, for $k \in \mathbb{C}$.

So let us just consider Case (*ii*) to determine when this occurs. Recall $\lim\limits_{z\to\infty} f(z) < \infty$ when $E = 0$, and $\lim\limits_{z\to\infty} g(z) = \infty$ when $E \neq 0$ or $F \neq 0$. Therefore,

$$R^{-1}(\infty) = \{\infty\} \text{ when } E = 0 \text{ and } F \neq 0.$$

Notice that Case (*iii*) cannot occur, since $\lim\limits_{z\to\infty} f(z) = \infty$ when $E \neq 0$ but $\lim\limits_{z\to\infty} g(z) < \infty$ only when $E = F = 0$.

Case (*iv*) may be excluded, since the sum/difference of two finite complex values is not infinite; therefore, $R^{-1}(\infty) \neq \{\infty\}$.

Thus, when $\overline{\deg}(r) \neq \underline{\deg}(r) + 1$, the only condition for the coefficients in (3.1) that must be satisfied for $R^{-1}(\infty) = \{\infty\}$ is when $E = 0$ and $F \neq 0$.

Therefore, we have shown the only such $w$ that will satisfy $R^{-1}(w) = \{w\}$ are the finite roots and poles, and $\infty$, if $E = 0$ and $F \neq 0$ for $E, F$ defined in (3.1), when $\overline{\deg}(r) \neq \underline{\deg}(r) + 1$.

The rest of the proof will continue by using Lemma 3.1.1 to consider all rational functions, $r(z)$, whose Newton map, $R(z)$, is of degree two. As we have mentioned, this separates all considered $r(z)$ into two main cases: (1) $\overline{\deg}(r) \neq \underline{\deg}(r) + 1$ and (2) $\overline{\deg}(r) = \underline{\deg}(r) + 1$. Then, in each case we will verify for which conditions, if any, that (3.3) holds true. If so, we will verify that $R^{-1}(w) = \{w\}$ can or cannot hold for $w$ defined as a root or pole. In addition, for the first main case, when $\overline{\deg}(r) \neq \underline{\deg}(r) + 1$ is satisfied, we must verify $E = 0$ and $F \neq 0$ in the Newton map (3.1). Furthermore, if equality occurs for a finite or infinite fixed point, then $R(z)$ is conjugate to a quadratic polynomial, and the Möbius transformation, $M(z)$, such that $MRM^{-1}(z) = z^2 + c$ will be constructed with $c$ given explicitly.

Let $r(z) = \dfrac{p(z)}{q(z)}$, where $\deg(p) = d$, $\deg(q) = e$, $p(z)$ has $n$ distinct roots and $q(z)$ has $m$ distinct roots. In addition, through all cases $r_i, p_i \in \mathbb{C}$, $d_i, e_i \in \mathbb{Z}_+$ for all $i$, and all roots and poles that are written are distinct, i.e. $r_i \neq p_j$, $r_i \neq r_j$ and $p_i \neq p_j$ for all $i, j$.

Case (1) $\deg(p) \neq \deg(q) + 1$ $(n + m = 2)$.

Case (1a) $n = 2, m = 0$.

Let $r(z) = (z - r_1)^{d_1}(z - r_2)^{d_2}$

By a straightforward calculation, the Newton map of $r(z)$ is of the form (3.1), where:

$$\begin{aligned} A &= 1 - d_1 - d_2, \\ B &= d_1 r_2 + d_2 r_1 - r_1 - r_2, \\ C &= r_1 r_2, \\ D &= 0, \\ E &= -(d_1 + d_2), \\ F &= d_1 r_2 + d_2 r_1. \end{aligned}$$

We will first determine if there are conditions for which $R^{-1}(r_i) = \{r_i\}$ for $i \in \{1, 2\}$. Simplifying (3.3) by using the above conditions where $z := r_i$, for $i \in \{1, 2\}$ we obtain:

$$(r_1 - r_2)^2(d_i - 1)^2 = 0. \tag{3.10}$$

Notice that this can only be true if $d_i = 1$ for $i = 1$ or 2, since the roots are unique. Therefore, for there to exist a finite $w$ such that $R^{-1}(w) = \{w\}$ at least one root of $r(z) = (z - r_1)^{d_1}(z - r_2)^{d_2}$ must be simple. Hence, the Newton map of

$$r(z) = (z - r_1)(z - r_2)^d$$

is conjugate to a quadratic polynomial.

Recall again from Lemma 6.1.4, since $\overline{\deg}(r) \neq \underline{\deg}(r) + 1$, the Newton map also has an infinite fixed point. For this reason we must check whether $E = 0$ and $F \neq 0$ to determine *all* conditions on the multiplicity of the roots of $r(z)$ that provide a $w$ such that $R^{-1}(w) = \{w\}$. However, $E = -(d_1 + d_2) < 0$, since $d_i$ is the multiplicity of the root $r_i$; therefore, $E \neq 0$ implies $R^{-1}(\infty) \neq \{\infty\}$.

To summarize Case (1$a$) so far, we have found the only $w \in \mathbb{C}_\infty$ which satisfy $R^{-1}(w) = \{w\}$ are the $w = r_i$ for $i \in \{1, 2\}$, such that $d_i = 1$. Without loss of generality, suppose $d_1 = 1$ and $d_2 = d$. We now wish to construct the Möbius transformation, $M(z)$, that conjugates $R(z)$ to $z^2 + c$. To do so we follow a similar process as in the proof of Theorem 2.10.1.

Suppose $M(z) = \dfrac{\alpha z+\beta}{\gamma z+\delta}$, and let $r(z) = (z-r_1)(z-r_2)^d$, which yields:

$$R(z) = \frac{-dz^2 + r_1(d-1)z + r_1 r_2}{-(1+d)z + r_1 d + r_2}.$$

To conjugate $R(z)$ we simplify

$$MRM^{-1}(z) = \frac{A'z^2 + B'z + C'}{D'z^2 + E'z + F'},$$

where:

$$\begin{aligned}
A' &= \alpha r_1\delta\gamma(1-d) + \beta\gamma\delta(1+d) + \beta\gamma^2(r_2 + dr_1) + \alpha\gamma^2 r_1 r_2 - \alpha\delta^2 d,\\
B' &= -(r_1\alpha + \beta)(2\alpha\gamma r_2 + \alpha\delta - \alpha\delta d + \gamma\beta + \gamma d\beta),\\
C' &= \alpha(\beta + \alpha r_1)(\beta + \alpha r_2),\\
D' &= \gamma(\delta + \gamma r_1)(\delta + \gamma r_2),\\
E' &= -(\delta + r_1\gamma)(2\alpha\gamma r_2 + \alpha\delta + \alpha\delta d + \gamma\beta - \gamma\beta d),\\
F' &= -\beta^2\gamma d + \alpha^2\delta d r_1 + \alpha^2\gamma r_1 r_2 - \alpha\beta\gamma d r_1 + \alpha^2\delta r_2 + \alpha\beta\gamma r_1 + \alpha\beta\delta + \alpha\beta\delta d.
\end{aligned}$$

Note, for $MRM^{-1}(z) = z^2 + c$, we must have $B' = D' = E' = 0$ and $\dfrac{A'}{F'} = 1$. Thus, solving for $\alpha, \beta, \gamma$ and $\delta$ in these four equations we obtain:

$$\begin{aligned}
\alpha &= \frac{-\delta(d+1)}{2dr_1},\\
\beta &= -\frac{-\delta(r_1 - r_2) + \alpha d r_1^2}{dr_1},\\
\gamma &= \frac{-\delta}{r_1},\\
\delta &= \delta.
\end{aligned}$$

Hence, after simplifying, the Möbius transformation becomes

$$M(z) = \frac{\alpha z+\beta}{\gamma z+\delta} = \frac{-z(1+d) - r_1 + 2r_2 + r_1 d}{2d(r_1 - z)}.$$

Thus, computing with these new values,

$$MRM^{-1}(z) = z^2 + c = z^2 + \frac{1}{4} - \frac{1}{4d^2}.$$

Note, since $r_1 \neq r_2$, it is true that $2dr_1(1-d) \neq 2d(r_1 - 2r_2 + r_1 d)$; therefore $M(z)$ is a Möbius transformation.

Hence we have shown that the Newton map of a polynomial of the form, $r(z) = (z - r_1)(z - r_2)^d$, is conjugate to $z^2 + \frac{1}{4} - \frac{1}{4d^2}$. This is Case ($i$) in the statement of the theorem. In addition, the case where $c = 0$ (if and only if $d = 1$) is well known and will be discussed in detail in §7.2.

Case (1b) $n = 1, m = 1.$

Let $r(z) = \dfrac{(z-r_1)^d}{(z-p)^e}$, such that $d \neq e+1$. The Newton map of $r(z)$ is of the form (3.1), where:

$$\begin{aligned} A &= 1 + e - p, \\ B &= pd - r_1 - p - r_1 e, \\ C &= r_1 p, \\ D &= 0, \\ E &= e - d, \\ F &= er_1 - dp. \end{aligned}$$

We will first examine if there are conditions for which $R^{-1}(r_1) = \{r_1\}$. Letting $z = r_1$ in (3.3) and simplifying with the above variables we obtain:

$$(p - r_1)^2(d-1)^2 = 0.$$

This implies that $r_1$ must be a simple root, i.e. $d = 1$, since $p \neq r_1$.

Now let us examine if there are conditions for which $R^{-1}(p) = \{p\}$. Letting $z = p$ in (3.3) and simplifying we obtain:

$$(p - r_1)^2(e+1)^2 = 0.$$

Notice that this cannot occur, since $p \neq r_1$ and $e \in \mathbb{Z}_+$.

Finally, let us examine if there are conditions for which $R^{-1}(\infty) = \{\infty\}$. As stated previously, this is only true if $E = 0$ and $F \neq 0$, which can occur if $d = e$.

Now to find the Möbius transformations, for both cases, $r_1$ a simple root and then for $d = e$.

First let us suppose $r_1$ is a simple root. We are considering the rational function $r(z) = \dfrac{z - r_1}{(z-p)^d}$. By the same process as in Case $(1a)$ we obtain the Möbius transformation, $M(z) = \dfrac{z(1-d) + (r_1 + dr_1 - 2p)}{2d(r_1 - z)}$, such that $MRM^{-1}(z) = z^2 + c$ where $c = \dfrac{1}{4} - \dfrac{1}{4d^2}$. Note, $M(z)$ is a Möbius transformation, since $r_1 \neq p$. This is Case $(iii)$ in the statement of the theorem.

Consider the second case of the original rational function $r(z) = \dfrac{(z-r_1)^d}{(z-p)^d}$, by writing the Newton map, $R(z) = \dfrac{z^2 - (r_1 + p + d(r_1 - p))z + r_1 p}{d(p - r_1)}$. We see that it is a polynomial, so we may simply apply Theorem 2.10.1. Thus, $R(z)$ is conjugate to the polynomial $z^2 + \dfrac{1}{4} - \dfrac{1}{4d^2}$ by

$$M(z) = \frac{-1}{d(r_1 - p)} z + \frac{r_1 + p + d(r_1 - p)}{2d(r_1 - p)}.$$

This is Case $(ii)$ in the statement of the theorem.

Note the case where $d = 1 = e$ will be discussed in detail in §7.1.

Case (1c) $n = 0, m = 2.$

Let $r(z) = \dfrac{1}{(z-p_1)^{e_1}(z-p_2)^{e_2}}$. By a simple calculation we obtain the Newton map of the form (3.1), where:

$$\begin{aligned} A &= -(e_1 + e_2 + 1), \\ B &= p_1 + p_2 + p_1 e_2 + p_2 e_1, \\ C &= -p_1 p_2, \\ D &= 0, \\ E &= -(e_1 + e_2), \\ F &= p_1 e_2 + p_2 e_1. \end{aligned}$$

We will examine under what conditions, if at all, is $R^{-1}(p_i) = \{p_i\}$, for $i \in \{1, 2\}$. Simplifying (3.3) by using the above conditions where $z := p_i$, for $i \in \{1, 2\}$ we obtain:

$$(p_1 - p_2)^2 (e_i + 1)^2 = 0.$$

Notice, this cannot occur, since the poles are unique, and $e_i$ is the multiplicity of the pole $p_i$, hence, a positive integer.

Since the Newton map also has an infinite fixed point we must verify if $E = 0$ and $F \neq 0$ can occur. Notice that $E = -(e_1 + e_2) \neq 0$; therefore, $R^{-1}(\infty) \neq \{\infty\}$.

Hence, the Newton map of any rational function of the form $r(z) = \dfrac{1}{(z-p_1)^{e_1}(z-p_2)^{e_2}}$ is not conjugate to a quadratic polynomial, since there do not exist any points such that $R^{-1}(w) = \{w\}$. We describe some interesting results considering this form in Chapter 10

At this point we have considered all rational functions, $r(z)$ such that $\overline{\deg}(r) \neq \underline{\deg}(r) + 1$, and now will consider the case of equality. Notice the Newton maps of the following functions do not have an infinite fixed point. Therefore, the only equality we must verify is equation (3.3) for $z$ defined as the finite roots and poles of the original function, $r(z)$.

Case (2) Let $\deg(p) = \deg(q) + 1$ $(n + m = 3)$.

Case (2a) $n = 3, m = 0$.

Let $r(z) = (z - r_1)^{d_1}(z - r_2)^{d_2}(z - r_3)^{d_3}$ . The Newton map of this $r(z)$ will not be of degree 2 by Lemma 3.1.1, because $d_1 + d_2 + d_3 = \deg(p) \neq \deg(q) + 1 = 0 + 1$, since $d_i \in \mathbb{Z}_+$ for $i = 1, 2, 3$. Therefore, since $\deg(R) \neq 2$, the Newton map cannot be conjugated to a quadratic polynomial (by Proposition 2.5.4).

Case (2b) $n = 2, m = 1$.

Let $r(z) = \dfrac{(z - r_1)^{d_1}(z - r_2)^{d_2}}{(z - p)^e}$, where $d_1 + d_2 = e + 1$. By substituting $e = d_1 + d_2 - 1$ we obtain the Newton map of the form (3.1), where:

$$\begin{aligned}
A &= p(1 - d_1 - d_2) + d_1 r_1 + d_2 r_2,\\
B &= -[r_1 p(1 - d_2) + r_2 p(1 - d_1) + r_1 r_2 (d_1 + d_2)],\\
C &= r_1 r_2 p,\\
D &= 1,\\
E &= -[r_1(1 - d_1) + r_2(1 - d_2) + p(d_1 + d_2)],\\
F &= p(r_1 d_2 + r_2 d_1) + r_1 r_2 (1 - d_1 - d_2).
\end{aligned}$$

We will first determine if $R^{-1}(r_i) = \{r_i\}$ for $i \in \{1, 2\}$ can occur, and if so, under what conditions. Simplifying (3.3) by using the above conditions where $z := r_i$, for $i \in \{1, 2\}$ we obtain:

$$(r_1 - r_2)^2(p - r_i)^2(d_i - 1)^2 = 0.$$

Notice that this equality holds if and only if $d_i = 1$, since the roots and poles are unique.

Now, Simplifying (3.3) by using the above conditions where $z := p$ we obtain:

$$(p - r_2)^2(p - r_1)^2(d_1 + d_2)^2 = 0,$$

which can never occur, due to our restrictions on the roots, poles and multiplicities.

Therefore, $R^{-1}(w) = \{w\}$ only when $d_1 = 1$ and $d_2 = e = d$, relabeling roots when needed. In conclusion we know that the Newton map of

$$r(z) = \frac{(z - r_1)(z - r_2)^d}{(z - p)^d}$$

is conjugate to a quadratic polynomial.

By repeating a similar process as in Case ($1a$), one constructs the $M(z)$ and $c$ as stated in the theorem under Case ($iv$). Note, $M(z)$ is a Möbius transformation, since $r_1 \neq p$. Note, the case where $d = 1$ will be the last special case explored in detail and will be done so in Chapter 8.

Case (2c) $n = 1, m = 2$.

Let $r(z) = \dfrac{(z - r_1)^d}{(z - p_1)^{e_1}(z - p_2)^{e_2}}$, such that $d = e_1 + e_2 + 1$. Using the fact that $d = e_1 + e_2 + 1$ we can write the Newton map of the form (3.1), where:

$$\begin{aligned}
A &= e_1(r_1 - p_1) + e_2(r_1 - p_2) + r_1,\\
B &= p_1p_2(e_1 + e_2) - r_1p_1(1 + e_2) - r_1p_2(1 + e_1),\\
C &= r_1p_1p_2,\\
D &= 1,\\
E &= e_1(r_1 - p_1) + e_2(r_1 - p_2) - p_1 - p_2,\\
F &= p_1p_2(1 + e_1 + e_2) - r_1(e_1p_2 + e_2p_1).
\end{aligned}$$

We will first simplify (3.3) by using the above conditions where $z := r_1$ we obtain:

$$(p_2 - r_1)^2(p_1 - r_1)^2(e_1 + e_2)^2 = 0,$$

which cannot occur, with our given restrictions on the roots poles and multiplicities.

Now evaluating (3.3) for $z := p_i$ for $i \in \{1, 2\}$ we obtain:

$$(p_i - r_1)^2(p_1 - p_2)^2(e_i + 1)^2 = 0,$$

which again, cannot occur with our given restrictions.

Therefore, $R^{-1}(w) \neq \{w\}$ where $R(z)$ is the Newton map of any function of the form $r(z) = \dfrac{(z - r_1)^d}{(z - p_1)^{e_1}(z - p_2)^{e_2}}$. Hence this Newton map is not conjugate to a quadratic polynomial.

Case (2d) $n = 0$, $m = 3$.

Let $r(z) = \dfrac{1}{(z - p_1)^{e_1}(z - p_2)^{e_2}(z - p_3)^{e_3}}$. The Newton map of this $r(z)$ will not be of degree 2, since $0 \neq e_1 + e_2 + e_3 + 1$, and we must have that $\deg(p) = \deg(q) + 1$ by Lemma 3.1.1.

In conclusion, examining all cases we conclude that there are only four possible rational functions in which the Newton map, $R(z)$, is conjugate to $z^2 + c$. One rational function was found in Case ($1a$), two were found in Case ($1b$) and one was found in Case ($2b$).

□

Theorem 3.1.4 provides much more than the rational functions whose Newton maps are conjugate to quadratic polynomials. Notice, that in every case $c$ satisfies $0 \leq c < \frac{1}{4}$, and recall from §2.11 that $\mathscr{M} \cup \mathbb{R} = \left[-2, \frac{1}{4}\right]$, where $\mathscr{M}$ is the Mandelbrot set. This fact, along with the equivalent definition of the Mandelbrot set implies the following.

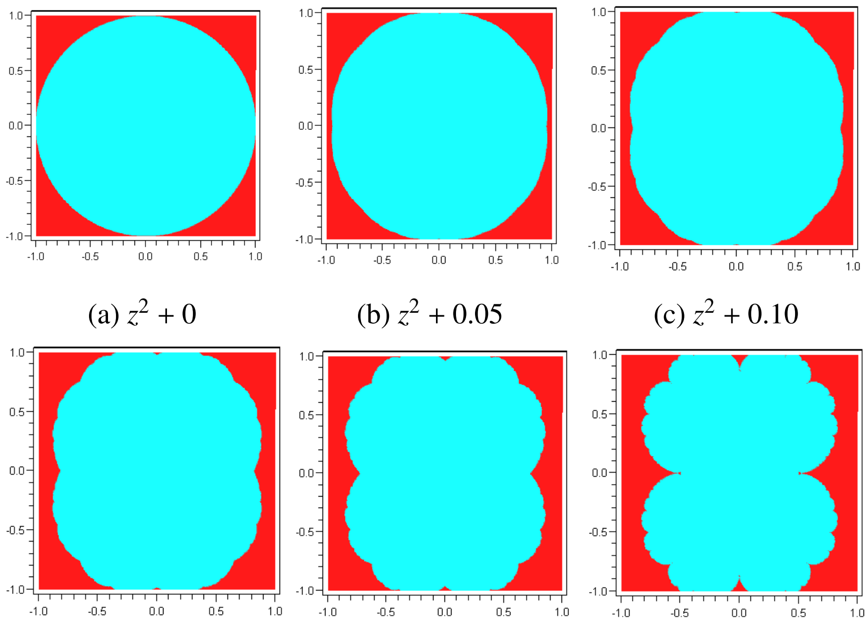
**Corollary 3.1.5.** *The Julia set of all Newton maps which are conjugate to a quadratic polynomial are connected.*

This corollary may not be impressive at first glance, since it is true that the Julia set of the Newton map of all polynomials is connected. However, it is not known the Julia set of the Newton map of rational functions is connected. This connectedness property is proved using the result of Shishikura which has been known since 1990 but was published in 2009 [39]. This result states that if a rational function of degree greater than one has exactly one repelling fixed point or indifferent fixed point then its Julia set is connected. Shishikura's result can be applied to the Newton map of polynomials since these rational functions have only one repelling fixed point at $\infty$, and all over fixed points are attracting. As Lemma 3.1.2 shows the Newton map of rational functions does not necessarily have exactly one repelling fixed point or indifferent fixed point. While this does not state the Julia set is not connected, this does show that one cannot use the same methods to prove the connectedness for all rational functions.

In addition, Theorem 3.1.4, also gives a detailed description the basins of attraction of the Newton map. This is due to the fact that once the multiplicities of the roots and poles are known, the exact value of $c$ is known, and there are many tools available to determine the basin of attraction of $z^2 + c$, for known $c$.

In fact, if $0 \leq c < \frac{1}{4}$ the Julia set of $z^2 + c$ resembles a wrinkly circle. See Figure 3.1 to view how the Julia set transforms as $c \rightarrow \frac{1}{4}$. As $c$ moves away from 0 toward $\frac{1}{4}$ the Julia set " 'crumples' locally, then bigger folds appear gradually" to quote Mandelbrot [27]. As in §2.9 the basin of attraction of the finite fixed point is the teal region centered around 0 and the basin of attraction of $\infty$ is the red region in Figure 3.1. The boundary separating the two regions is the Julia set.

While the Newton map of the rational functions stated in Theorem 3.1.4 do not have attracting fixed points at 0 and $\infty$ they do have two attracting fixed points, and one repelling fixed point. Therefore, while the size and location may vary, the teal region in Figure 3.1 is of the same shape as the basin of attraction of one of the fixed points of the Newton map, the same holds true for the red region, and the Julia set.



(a) $z^2 + 0$ (b) $z^2 + 0.05$ (c) $z^2 + 0.10$

(d) $z^2 + 0.15$ (e) $z^2 + 0.20$ (f) $z^2 + 0.25$

Figure 3.1: Basins of Attraction of $z^2 + c$ for $0 \leq c < \frac{1}{4}$.

If $c = 0$ we can state explicitly the form of the Julia set of the Newton map. These Julia sets will be investigated in detail in Chapters 7 and 8; however, we have summarized all the information within the following theorem.

**Theorem 3.1.6.** *Let $d = 1$ in the four cases of $r(z)$ in Theorem 3.1.4, then $R(z) \sim z^2$. In addition, the Julia sets of the cases are as follows:*

$$(i) \qquad J(R) = \begin{cases} \left\{ z = x + iy \;\middle|\; y = \dfrac{(r_2 - r_1)_x}{(r_1 - r_2)_y} x + \dfrac{|r_1|^2 - |r_2|^2}{2(r_1 - r_2)_y} \right\} & \text{if} \quad (r_1)_y \neq (r_2)_y \\ \left\{ z = x + iy \;\middle|\; x = \dfrac{(r_1)_x + (r_2)_x}{2} \right\} & \text{if} \quad (r_1)_y = (r_2)_y, \end{cases}$$

*$(ii)$ and $(iii)$* $J(R) = \{z \mid |z - r_1| = |r_1 - p|\}$

$$(iv) \qquad J(R) = \begin{cases} \left\{ re^{i\theta} + C \;\middle|\; \theta \in [0, 2\pi) \right\} & \text{if } |p - r_1| \neq |p - r_2|, \\ \left\{ z = x + iy \;\middle|\; y = \dfrac{(r_1 - r_2)_x}{(r_2 - r_1)_y} x + p_y - p_x \dfrac{(r_1 - r_2)_x}{(r_2 - r_1)_y} \right\} & \\ & \text{if } |p - r_1| = |p - r_2| \text{ and } (r_1)_y \neq (r_2)_y, \\ \left\{ z = x + iy \;\middle|\; x = p_x \right\} & \text{if } |p - r_1| = |p - r_2| \text{ and } (r_1)_y = (r_2)_y, \end{cases}$$

*where the radius, r, and the center of the circle, C in the first case, are given by,*

$$r = \frac{|(r_1 - r_2)(r_1 - p)(r_2 - p)|}{\left|(r_1 - r_2)_x (r_1 + r_2 - 2p)_x + (r_1 - r_2)_y (r_1 + r_2 - 2p)_y\right|},$$

$$C = \frac{(r_1 - r_2)_x((r_1)_x (r_2)_x - p_x^2 - p_y^2) + (r_1)_y (r_2)_x ((r_1)_y - 2p_y) - (r_1)_x (r_2)_y ((r_1)_x - 2p_y)}{(r_1 - r_2)_x (r_1 + r_2 - 2p)_x + (r_1 - r_2)_y (r_1 + r_2 - 2p)_y} +$$
$$i \frac{(r_1 - r_2)_y((r_2)_y (r_1)_y - p_x^2 - p_y^2) - (r_1)_y (r_2)_x ((r_1)_y - 2p_x) + (r_1)_x (r_2)_y ((r_1)_x - 2p_x)}{(r_1 - r_2)_x (r_1 + r_2 - 2p)_x + (r_1 - r_2)_y (r_1 + r_2 - 2p)_y}.$$

*Proof.* The proof follows from Theorems 7.2.2, 7.1.2 and 8.1.2. □

# CHAPTER 4
# DEGREE THREE NEWTON MAPS

We would ideally like to use the same method to categorize all degree 3 Newton maps as we did in Chapter 3. However, to re-quote Cayley, "the cubic equation appears to present considerable difficulty" [10]. In the quadratic cases, when computing $R^{-1}\{r_i\}$ and $R^{-1}\{p_j\}$, we obtained a fully factored form (for example, see equation (3.10)), with around 2 or 3 factors. Observing this factored form is when we were able to place the conditions on the multiplicities of the roots and poles of $r(z)$ to determine if $R(z)$ is conjugate to a quadratic polynomial. However, when attempting to simplify $R^{-1}\{r_i\}$ and $R^{-1}\{p_j\}$ for $\deg(R) = 3$ we do not obtain a fully factored form. Therefore, using the same procedure for degree 3 Newton maps we are presented with "considerable difficulty".

However, we can use different results to state that no Newton map of a rational function is conjugate to $z^3$. Let us look at a few results before we prove this result. The following result was briefly mentioned in [1], but will be proved in detail here.

**Proposition 4.1.1.** *Let* $f \overset{\overrightarrow{M}}{\sim} g$. *Then* $w$ *is an attracting, repelling or indifferent fixed point of* $g(z)$ *if and only if* $M(w)$ *is an attracting, repelling or indifferent fixed point of* $f(z)$*, respectively.*

*Proof.* Recall, by Proposition 2.5.3, since $f \overset{\overrightarrow{M}}{\sim} g$, if $w$ is a fixed point of $g(z)$ then $M(z)$ is a fixed point of $f(z)$.

Also, recall $f \overset{\overrightarrow{M}}{\sim} g$ means

$$f(z) = MgM^{-1}(z).$$

If $F(z) = G(z)$ then $F'(z) = G'(z)$ for all $z$, specifically, $F'(M(w)) = G'(M(w))$.

Consider,

$$f'(z) = M'\left(gM^{-1}(z)\right) \cdot g'\left(M^{-1}(z)\right) \cdot \left(M^{-1}\right)'(z),$$

evaluating at $z = M(w)$ we obtain:

$$\begin{aligned} f'(M(w)) &= M'\left(gM^{-1}(M(w))\right) \cdot g'\left(M^{-1}(M(w))\right) \cdot \left(M^{-1}\right)'(M(w)) \\ &= M'(g(w)) \cdot g'(w) \cdot \left(M^{-1}\right)'(M(w)) \end{aligned}$$

and recall that $g(w) = w$, so

$$f'(M(w)) = M'(w) \cdot g'(w) \cdot \left(M^{-1}\right)'(M(w)). \tag{4.1}$$

Notice

$$\left(M^{-1}\right)'(M(w)) \cdot M'(w) = \frac{d}{dz}\left[M^{-1}(M(z))\right]\Big|_{z=w} = \frac{d}{dz}[z]\Big|_{z=w} = 1.$$

Therefore equation (4.1) becomes:

$$f'(M(w)) = g'(w).$$

This implies $|f'(M(w))| = |g'(w)|$. Thus we have our conclusion.

□

**Corollary 4.1.2.** *If* $f \overset{\overrightarrow{M}}{\sim} g$ *then* $f(z)$ *and* $g(z)$ *have the same number of attracting, repelling and indifferent fixed points.*

*Proof.* This follows directly from Propositions 2.5.3 and 4.1.1.

□

One must be careful, Corollary 4.1.2 does say that if $f(z)$ and $g(z)$ do not have the same number of attracting, repelling and indifferent fixed points then they are not conjugate. However, it does *not* say if $f$ and $g$ have the same number of attracting, repelling and indifferent fixed points then they are conjugate.

Therefore, if we can show that for $\deg(R) = 3$, $R(z)$ does not have the same number of attracting, repelling and indifferent fixed points as $z^3$ then we know that $R(z) \nsim z^3$, and this is the method we will use to prove the following theorem. Recall Lemma 3.1.2 states the types of fixed points that the Newton map of rational functions has.

We now have all the information we need to state and prove our result concerning degree 3 Newton maps.

**Theorem 4.1.3.** *There are no Newton maps of rational functions that are conjugate to* $z^3$.

*Proof.* Recall Proposition 2.5.1 says that if two functions are conjugate they must have the same degree. Therefore, the only Newton maps we must consider are those of degree 3. In addition, by Lemma 3.1.1 we have that following rational maps are the only functions that will produce a degree 3 Newton map:

$$(z-r_1)^{d_1}(z-r_2)^{d_2}(z-r_3)^{d_3} \quad \text{where} \quad d_1+d_2+d_3 \neq 1 \tag{4.2}$$

$$\frac{(z-r_1)^{d_1}(z-r_2)^{d_2}}{(z-p_1)^{e_1}} \quad \text{where} \quad d_1+d_2 \neq 1+e_1 \tag{4.3}$$

$$\frac{(z-r_1)^{d_1}}{(z-p_1)^{e_1}(z-p_2)^{e_2}} \quad \text{where} \quad d_1 \neq 1+e_1+e_2 \tag{4.4}$$

$$\frac{1}{(z-p_1)^{e_1}(z-p_2)^{e_2}(z-p_3)^{e_3}} \quad \text{where} \quad 0 \neq 1+e_1+e_2+e_3 \tag{4.5}$$

$$\frac{(z-r_1)^{d_1}(z-r_2)^{d_2}(z-r_3)^{d_3}}{(z-p_1)^{e_1}} \quad \text{where} \quad d_1+d_2+d_3 = 1+e_1 \tag{4.6}$$

$$\frac{(z-r_1)^{d_1}(z-r_2)^{d_2}}{(z-p_1)^{e_1}(z-p_2)^{e_2}} \quad \text{where} \quad d_1+d_2 = 1+e_1+e_2 \tag{4.7}$$

$$\frac{(z-r_1)^{d_1}}{(z-p_1)^{e_1}(z-p_2)^{e_2}(z-p_3)^{e_3}} \quad \text{where} \quad d_1 = 1+e_1+e_2+e_3 \tag{4.8}$$

The following table shows the number of attracting and repelling fixed points for each of the above 7 functions. Note that if $\infty$ is always an attracting or repelling fixed point it is listed in one of the middle two columns. If $\infty$ is a fixed point, but the type depends on the multiplicities this data is listed in the last column. If $\infty$ is not a fixed point there will be dashes in the last column. The following abbreviations will be used in the table as well: afp-attracting fixed point; s-afp-super-attracting fixed point; rfp-repelling fixed point.

| Function | Number Attracting Fixed Points | Number Repelling Fixed Points | $\infty$ Case |
|---|---|---|---|
| (4.2) | 3 | 1 ($\infty$) | Always repelling |
| (4.3) | 2 | 1 | afp$\Leftrightarrow d_1+d_2<e_1$<br>s-afp$\Leftrightarrow d_1+d_2=e_1$<br>rfp$\Leftrightarrow d_1+d_2>e_1+1$ |
| (4.4) | 1 | 2 | afp$\Leftrightarrow d_1<e_1+e_2$<br>s-afp$\Leftrightarrow d_1=e_1+e_2$<br>rfp$\Leftrightarrow d_1>e_1+e_2+1$ |
| (4.5) | 1 ($\infty$) | 3 | Always attracting |
| (4.6) | 3 | 1 | — |
| (4.7) | 2 | 2 | — |
| (4.8) | 1 | 3 | — |

Now let us investigate the type of and number of fixed points of $z^3$. Notice $z^3$ has two super-attracting fixed points, $\{0, \infty\}$, and two repelling fixed points, $\{-1, 1\}$. The table above shows the only three functions that possibly fit this description are functions (4.3), (4.4) and (4.7). Therefore, by Proposition 4.1.1, if these three functions do not have 2 super-attracting fixed points and 2 repelling fixed points then they will not be conjugate to $z^3$

Recall by Theorem 6.1.1 and Theorem 6.1.4 the only finite super-attracting fixed points of a Newton map are the simple roots of the original function. In addition, if $\overline{\deg}(r) = \underline{\deg}(r)$ then $\infty$ is also a super-attracting fixed point, and if $\overline{\deg}(r) > \underline{\deg}(r) + 1$ then $\infty$ is a repelling fixed point.

Let us continue and look at each of the three functions to see if the Newton map can have two super-attracting fixed points:

(4.3) The two roots $r_1, r_2$ are super-attracting fixed points only if both are simple roots, in other words: $d_1 = d_2 = 1$. The only finite repelling fixed point is $p_1$, and $\infty$ is a repelling fixed point only if $d_1 + d_2 > e_1 + 1$, which cannot occur, since $1 + 1 = 2 \ngtr e_1 + 1$ where $e_1 \in \mathbb{Z}_+$. Therefore, the Newton map of function (4.3) cannot be conjugate to $z^3$.

(4.4) The root $r_1$ is super-attracting only if $d_1 = 1$. There are two finite repelling fixed points $p_1, p_2$. And $\infty$ is a super-attracting fixed point only if $d_1 = e_1 + e_2$, which cannot occur, since $1 \neq e_1 + e_2$ where $e_i \in \mathbb{Z}_+$, for $i = 1, 2$. Therefore, the Newton map of function (4.4) cannot be conjugate to $z^3$.

(4.7) The two roots $r_1, r_2$ are super-attracting when $d_1 = d_2 = 1$. There are two finite repelling fixed points at $p_1$ and $p_2$. In addition, for function (4.7) to have a degree 3 Newton map the powers must satisfy $d_1 + d_2 = e_1 + e_2 + 1$; however, this cannot occur since $1 + 1 = 2 \neq e_1 + e_2 + 1$ where $e_i \in \mathbb{Z}_+$ for $i = 1, 2$. Therefore, the Newton map of function (4.7) cannot be conjugate to $z^3$.

Hence, out of all the degree 3 Newton maps, none have 2 super-attracting and 2 repelling fixed points. Since $z^3$ has 2 super-attracting and 2 repelling fixed points this means that no Newton map is conjugate to $z^3$ by Proposition 4.1.2.

□

# CHAPTER 5
# THE DEGREE OF THE NEWTON MAP

We now seek to discuss the properties a rational function and how those properties relate to the degree of its Newton map. Recall, this result was stated in Chapter 3; however, we will state this again and prove it here. Our first result extends the degree of the Newton map of any rational function not just the Newton map of a polynomial. Then we will state the known result of the degree of the Newton map of polynomials and show how this relates to our theorem.

Recall, if $r(z) = \frac{p(z)}{q(z)}$, for $p, q$ coprime polynomials then $\overline{\deg}(r) = \deg(p)$, and $\underline{\deg}(r) = \deg(q)$.

**Lemma 3.1.1.** *Let $r(z)$ be a reduced rational function such that $\overline{\deg}(r) = d$, $\underline{\deg}(r) = e$. Let $n \leq d$ and $m \leq e$ be the number of distinct roots and poles, respectively. Let $R(z)$ be the Newton map of $r(z)$, then*

$$\deg(R) = \begin{cases} n+m & \textit{if } d \neq e+1 \\ n+m-1 & \textit{otherwise.} \end{cases}$$

Notice the degree of the Newton map is the sum (or one less than the sum) of the number of unique roots and poles, and not the sum (or less than the sum) of the degree of the numerator and denominator.

*Proof.* Let $r(z) = \frac{p(z)}{q(z)}$ where $p(z) = \prod_{i=1}^{n}(z-r_i)^{d_i}$, $q(z) = \prod_{j=1}^{m}(z-p_j)^{e_j}$, $r_i \neq p_j$ for all $i, j$, $\sum_{i=1}^{n} d_i = d$ and $\sum_{j=1}^{m} e_j = e$. We will now proceed by using the logarithmic derivative.

Consider the following:

$$\begin{aligned} \log(r(z)) &= \textstyle\sum_{i=1}^{n} d_i \log(z-r_i) - \sum_{j=1}^{m} e_j \log(z-p_j), \\ \frac{r'(z)}{r(z)} &= \frac{d}{dz}\log(r(z)) \\ &= \textstyle\sum_{i=1}^{n} \dfrac{d_i}{z-r_i} - \sum_{j=1}^{m} \dfrac{e_j}{z-p_j} \\ &= \frac{\sum_{i=1}^{n}\prod_{\substack{k=1\\k\neq i}}^{n} d_i(z-r_k)\prod_{\ell=1}^{m}(z-p_\ell) - \sum_{j=1}^{m}\prod_{\substack{\ell=1\\\ell\neq j}}^{m} e_j(z-p_\ell)\prod_{k=1}^{n}(z-r_k)}{\prod_{k=1}^{n}(z-r_k)\prod_{\ell=1}^{m}(z-p_\ell)}. \end{aligned}$$

The Newton map is of the form

$$\begin{aligned}
R(z) &= z - \frac{r(z)}{r'(z)} \\
&= z - \frac{1}{\frac{d}{dz}\log(r(z))} \\
&= z - \frac{\prod_{k=1}^{n}(z-r_k)\prod_{\ell=1}^{m}(z-p_\ell)}{\sum_{i=1}^{n}\prod_{\substack{k=1\\k\neq i}}^{n} d_i(z-r_k)\prod_{\ell=1}^{m}(z-p_\ell) - \sum_{j=1}^{m}\prod_{\substack{\ell=1\\\ell\neq j}}^{m} e_j(z-p_\ell)\prod_{k=1}^{n}(z-r_k)}.
\end{aligned}$$

For convenience let

$$g(z) = \sum_{i=1}^{n}\prod_{\substack{k=1\\k\neq i}}^{n} d_i(z-r_k)\prod_{\ell=1}^{m}(z-p_\ell) \text{ and } h(z) = \sum_{j=1}^{m}\prod_{\substack{\ell=1\\\ell\neq j}}^{m} e_j(z-p_\ell)\prod_{k=1}^{n}(z-r_k),$$

thus we my rewrite the Newton map in the form:

$$R(z) = \frac{zg(z) - zh(z) - \prod_{k=1}^{n}(z-r_k)\prod_{\ell=1}^{m}(z-p_\ell)}{g(z)-h(z)}.$$

There are five main components in the numerator and denominator that must be considered for determining the degree of $R(z)$. Breaking the numerator into three main terms and finding the degree and leading coefficient of each of these terms we see that:

| Term | Degree | Leading Coefficient |
|---|---|---|
| $zg(z)$ | $n+m$ | $\sum_{i=1}^{n} d_i = d$ |
| $-zh(z)$ | $n+m$ | $-\sum_{j=1}^{m} e_j = -e$ |
| $-\prod_{k=1}^{n}(z-r_k)\prod_{\ell=1}^{m}(z-p_\ell)$ | $n+m$ | $-1$ |

In addition, separating the denominator into its two main terms and looking at the degree and the leading coefficient of each we see:

| Term | Degree | Leading Coefficient |
|---|---|---|
| $g(z)$ | $n+m-1$ | $\sum_{i=1}^{n} d_i = d$ |
| $-h(z)$ | $n+m-1$ | $-\sum_{j=1}^{m} e_j = -e$ |

Therefore, the first table concludes that $\overline{\deg}(R) = n+m$, providing the coefficients of the $z^{n+m}$ term do not cancel, in other words $d-e-1 \neq 0$ or equivalently $d \neq e+1$. The second table concludes that $\underline{\deg}(R) = n+m-1$, providing $d-e \neq 0$ or equivalently $d \neq e$.

Since $n + m > n + m - 1$, we know if $d \neq e + 1$, that

$$\deg(R) = \max\{\overline{\deg}(R), \underline{\deg}(R)\} = \overline{\deg}(R) = n + m.$$

However, when $d = e + 1$ the coefficient of the $z^{n+m}$ term in the numerator becomes 0; therefore, $\overline{\deg}(R) \leq n + m - 1$. Notice $d = e + 1$ implies $d - e \neq 0$, the coefficient of the $z^{n+m-1}$ term in the denominator, so $\underline{\deg}(R) = n + m - 1$ even if $d = e + 1$. Hence, regardless of the new leading coefficients of the numerator, when $d = e + 1$,

$$\deg(R) = \max\{\overline{\deg}(R), \underline{\deg}(R)\} = \underline{\deg}(R).$$

Thus, $\deg(R) = n + m - 1$ when $d = e + 1$.

□

**Corollary 5.1.2.** *Let $P(z)$ be the Newton map of polynomial $p(z)$, where $n$ is the number of distinct roots of $p(z)$ then* $\deg(P) = n \leq \deg(p)$.

*Proof.* This follows directly from Theorem 3.1.1 where $q(z) \equiv 1$, which implies $m = e = 0$.

□

# CHAPTER 6
# THE FIXED POINTS OF THE NEWTON MAP

We will now state and prove the new results relating the finite roots and poles of the original rational function, $r(z)$, to its Newton map, $R(z)$. We will then consider $\infty$ as a fixed point of $R(z)$ and see what properties $r(z)$ must have for this to occur. Note, these fixed point properties are written in the literature of the Newton map applied to polynomials, but our literature search has not found information concerning the fixed points of a Newton map applied to rational functions. In addition, in Chapter 3 we stated the following results as one lemma, here we will separate the results in order to provide easier to follow proofs.

However, before we move onto the fixed points we will address $\infty$. Infinity is treated no differently than any other point, because of our use of the chordal metric which is defined below.

For $z, w \in \mathbb{C}_\infty$ the **chordal metric** is defined by:

$$\sigma(z, w) = \frac{2|z - w|}{(1 + |z|^2)^{1/2}\,(1 + |w|^2)^{1/2}},$$

$$\sigma(z, \infty) = \lim_{w\to\infty} \sigma(z, w) = \frac{2}{(1 + |z|^2)^{1/2}}.$$

Therefore, when considering $\infty$, the chordal metric on $\mathbb{C}_\infty$ will be used instead of the standard Euclidean metric on $\mathbb{C}$. Note, when using the chordal metric instead of the Euclidean metric the Julia and Fatou sets of the function $R(z)$ will not change, since $\{R^n\}_{n=0}^{\infty}$ is equicontinuous with respect to the Euclidean metric if and only if $\{R^n\}_{n=0}^{\infty}$ is equicontinuous with respect to the chordal metric. Thus, $\infty$ can, and will be, considered as any other point in the complex plane.

Now, onto the fixed points of the Newton map. Recall that a fixed point is a point $w$ such that $f(w) = w$.

**Lemma 6.1.1.** *Let $r(z)$ be a reduced rational function. The finite roots and poles of $r(z)$ are the only finite fixed points of the Newton map $R(z)$.*

*Proof.* Consider the original rational function $r(z) = \dfrac{p(z)}{q(z)}$ and its Newton map:

$$R(z) = z - \frac{p(z)q(z)}{p'(z)q(z) - p(z)q'(z)}.$$

Notice, $R(w) = w$ if and only if $p(w) = 0$, where $w$ is a root, or $q(w) = 0$, where $w$ is a pole of $r(z)$.

□

Note the roots of any function, not just a rational function, are fixed points of its Newton map.

We will introduce a definition here, as it relates to Lemma 6.1.2 A fixed point of the Newton map, $R(z)$, is an **extraneous fixed point** if it is a fixed point of $R(z)$ and yet not a root of the original rational function, $r(z)$. As with fixed points an extraneous fixed point may be attracting, repelling or indifferent. For example, let $r(z) = \dfrac{z^2 - 1}{z + 5i}$, then the Newton map is $R(z) = z - \dfrac{(z^2 - 1)(z + 5i)}{z^2 + 10iz + 1}$. Notice the fixed points of $R(z)$ are the roots and poles of the original rational function: $\pm 1, -5i$. While we call $\pm 1$ and $-5i$ the fixed points of $R(z)$, $-5i$ is, more specifically, called an extraneous fixed point of $R(z)$.

While Theorem 6.1.1 shows that the poles of the original function are extraneous fixed points of the Newton map, Theorem 6.1.2 will show that these extraneous fixed points are repelling.

It is well known that the roots of a polynomial are attracting fixed points of the Newton map, and this also follows from Theorem 6.1.2. However, to the author's knowledge the roots and poles of a rational function have not been addressed concerning the fixed points of the Newton map.

**Lemma 6.1.2.** *Let $r(z)$ be a reduced rational function, then the Newton map, $R(z)$, has attracting fixed points at the finite roots of $r(z)$, and repelling fixed points at the finite poles of $r(z)$. In addition, the super-attracting fixed points of $R(z)$ are the simple roots of $r(z)$.*

Before the proof we will discuss the importance of Lemma 6.1.2. It shows that while the Newton map has fixed points at the poles of the original function, these extraneous fixed points are not attracting, but rather repelling. Recall, by definition, for a fixed point, $w$, if $|R'(w)| > 1$ then $w$ is a repelling fixed point. For $z$ in a neighborhood of $w$ we may approximate $R'(w) \approx \dfrac{R(w) - R(z)}{w - z}$, which implies that for $w$, a repelling fixed point, $|R(w) - R(z)| > |w - z|$. Thus, points close to a pole of the original rational function will diverge away from the fixed point. Using this inequality one may prove the only way for $R^n(z) \to w$ is if there exists an $n$ such that $R^n(z) = w$. Therefore, even through $R(z)$ has extraneous fixed points at the poles, the Newton map will not converge to non-roots of the original function (that is unless $R^n(z) = w$ for some $n$ and $w$ an extraneous fixed point).

Note, this theorem will be proved by using the properties of the logarithmic derivative, as in the proof of Lemma 3.1.1.

*Proof.* Let $\{r_i\}_{i=1}^n$ be the roots and $\{p_j\}_{j=1}^n$ be the poles of $r(z)$. Lemma 6.1.1 states that the only finite fixed points of the Newton map, $R(z) = z - \frac{r(z)}{r'(z)}$, are contained in these sets. Recall, if $|R'(w)| < 1$, then $w$ is an attracting fixed point, and if $|R'(w)| > 1$ then $w$ is a repelling fixed point. In addition, if $R'(w) = 0$ then $w$ is a super-attracting fixed point.

Let

$$r(z) = \frac{p(z)}{q(z)} = \frac{\prod_{i=0}^n (z - r_i)^{d_i}}{\prod_{j=0}^m (z - p_j)^{e_j}},$$

with $r_i \neq p_j$ for all $i$ and $j$. Recall that

$$R(z) = z - \frac{r(z)}{r'(z)}.$$

Therefore we have the following,

$$\begin{aligned} |R'(z)| &= \left| 1 - \frac{d}{dz}\left(\frac{1}{\frac{r'(z)}{r(z)}}\right)\right| \\ \log(r(z)) &= \log(p(z)) - \log(q(z)) \\ &= \sum_{i=0}^{n} d_i \log(z - r_i) - \sum_{j=0}^{m} e_j \log(z - p_j). \end{aligned}$$

Therefore,

$$\frac{r'(z)}{r(z)} = \frac{d}{dz}\log(r(z)) = \sum_{i=0}^{n} \frac{d_i}{z - r_i} - \sum_{j=0}^{m} \frac{e_j}{z - p_j},$$

and an easy calculation shows,

$$\frac{d}{dz}\left(\frac{r(z)}{r'(z)}\right) = \frac{d}{dz}\left(\frac{1}{\frac{r'(z)}{r(z)}}\right) = \frac{\displaystyle\sum_{i=0}^{n} \frac{d_i}{(z - r_i)^2} - \sum_{j=0}^{m} \frac{e_j}{(z - p_j)^2}}{\left(\displaystyle\sum_{i=0}^{n} \frac{d_i}{z - r_i} - \sum_{j=0}^{m} \frac{e_j}{z - p_j}\right)^2}.$$

Since $r_i$ is a root of $r(z)$ of multiplicity $d_i \geq 1$, then

$$\begin{aligned}
|R'(r_i)| &= \lim_{z\to r_i} |R'(z)| \\
&= \lim_{z\to r_i} \left| 1 - \frac{\dfrac{d_i}{(z-r_i)^2} + \displaystyle\sum_{\substack{k=1\\k\neq i}}^{n} \frac{d_k}{(z-r_k)^2} - \sum_{j=0}^{m} \frac{e_j}{(z-p_j)^2}}{\left( \dfrac{d_i}{z-r_i} + \displaystyle\sum_{\substack{k=1\\k\neq i}}^{n} \frac{d_k}{z-r_k} - \sum_{j=0}^{m} \frac{e_j}{z-p_j} \right)^2} \right| \\
&= \left| 1 - \frac{d_i}{d_i^2} \right| \\
&= \left| 1 - \frac{1}{d_i} \right|. \qquad (6.1)
\end{aligned}$$

Since $d_i \geq 1$, equation (6.1) is strictly less than 1. Hence, $r_i$ is an attracting fixed point of $R(z)$. In addition, when $r_i$ is a simple root of $r(z)$ then $d_i = 1$ and by equation (6.1) we have that $|R'(r_i)| = |1 - 1| = 0$. Thus if $r_i$ is a (simple) root of $r(z)$ then $r_i$ is a (super-)attracting fixed point of $R(z)$.

Since $p_j$ is a pole of $r(z)$ of multiplicity $e_j \geq 1$. We see that,

$$\begin{aligned}
\left|R'(p_j)\right| &= \lim_{z\to p_j} |R'(z)| \\
&= \lim_{z\to p_j} \left| 1 - \frac{\displaystyle\sum_{i=0}^{n} \frac{d_i}{(z-r_i)^2} - \frac{e_j}{(z-p_j)^2} - \sum_{\substack{\ell=1\\ \ell\neq j}}^{m} \frac{e_\ell}{(z-p_\ell)^2}}{\left( \displaystyle\sum_{i=0}^{n} \frac{d_i}{z-r_i} - \frac{e_j}{z-p_j} - \sum_{\substack{\ell=1\\ \ell\neq j}}^{m} \frac{e_\ell}{z-p_\ell} \right)^2} \right| \\
&= \left| 1 - \frac{-e_j}{e_j^2} \right| \\
&= \left| 1 + \frac{1}{e_j} \right|. \qquad (6.2)
\end{aligned}$$

Therefore, since $e_j \geq 1$, equation (6.2) is greater then 1. Hence, the poles of $r(z)$ are repelling fixed points of $R(z)$.

□

It is known that $\infty$ is always a repelling fixed point for the Newton map of polynomials; however, the type of fixed point $\infty$ is for the Newton map of rational functions has not been stated, and this can be found in Lemma 6.1.4. Although, before we state this result will need a simple lemma.

**Lemma 6.1.3.** *Let* $F(z) = \dfrac{f(z)}{g(z)}$ *be a rational function such that* $\overline{\deg}(F) = a$ *and* $\underline{\deg}(F) = b$. *If the leading coefficient of the numerator of* $F'(z)$ *is nonzero (i.e.,* $lc(f)lc(g)(a-b) \neq 0$*) then*

$$\begin{aligned} \overline{\deg}(F') &> \underline{\deg}(F') \quad \textit{when } a > b+1,\\ \overline{\deg}(F') &< \underline{\deg}(F') \quad \textit{when } a < b+1,\\ \overline{\deg}(F') &= \underline{\deg}(F') \quad \textit{when } a = b+1. \end{aligned}$$

The proof of Lemma 6.1.3 is straightforward and will not be included.

**Lemma 6.1.4.** *For the Newton map,* $R(z)$*, of a rational function,* $r(z)$*, where* $\overline{\deg}(r) = d$*, and* $\underline{\deg}(r) = e$*,* $\infty$ *is a fixed point if and only if* $d \neq e+1$. *Moreover, in this case,*

$$\infty \textit{ is } = \begin{cases} \textit{a super-attracting fixed point} & \textit{for} \quad d = e,\\ \textit{an attracting fixed point} & \textit{for} \quad d < e,\\ \textit{a repelling fixed point} & \textit{for} \quad d > e+1,\\ \textit{an indifferent fixed point} & \textit{never.} \end{cases}$$

Before we proceed to the proof, we provide a word of caution about multiple roots and poles. If $r(z)$ has a multiple root (pole) then it corresponds to a factor of both $p(z)$ and $p'(z)$ ($q(z)$ and $q'(z)$). From the definition of $R(z)$ it is then clear that the linear factor corresponding to the multiple root (pole) can be factored and canceled in both the numerator and denominator, which decreases the degree of $R(z)$. For this reason, in Lemma 3.1.1, to sufficiently take multiplicities into consideration, we use the factored version of $p(z)$ and $q(z)$ as well as logarithms. However, for Lemma 6.1.4 we are *comparing* the degrees of the numerator and denominator, i.e. we are asking, are the degrees of the numerator and denominator equal, or which is greater, not stating the exact degree. Therefore, while a multiple root (pole) of $r(z)$ will decrease the degree of $R(z)$, it will decrease the degree of both the numerator and denominator by the same amount, and the (in)equality between the degree of the numerator and denominator will still hold.

For simplicity, only in the proof of Lemma 6.1.4 we will state the degree of the numerator and denominator as an exact value. We acknowledge that the degree may be less than this value, depending on multiple roots (poles). However, since the degree will decrease by the same amount in the numerator and denominator this does not affect our (in)equalities.

*Proof.* Let

$$\begin{aligned}
p(z) &= z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0, \\
q(z) &= z^e + b_{e-1}z^{e-1} + \cdots + b_1 z + b_0, \\
r(z) &= \frac{p(z)}{q(z)}, \\
R(z) &= z - \frac{r(z)}{r'(z)} \\
&= z - \frac{p(z)q(z)}{q(z)p'(z) - p(z)q'(z)} \\
&= \frac{zq(z)p'(z) - zp(z)q'(z) - p(z)q(z)}{q(z)p'(z) - p(z)q'(z)}, \\
R'(z) &= 1 - \frac{(q(z)p'(z))^2 - (p(z)q'(z))^2 - p(z)q(z)(p''(z)q(z) - p(z)q''(z))}{(q(z)p'(z) - p(z)q'(z))^2}.
\end{aligned}$$

Clearly $\infty$ is a fixed point of $R(z)$ if and only if $\overline{\deg}(R) > \underline{\deg}(R)$. Therefore, we must determine the degree of the numerator and denominator. To do so we will look at the degree of each of the main terms within the numerator and denominator, as we did in the proof of Theorem 3.1.1. Notice we have the following degrees:

Numerator:

$$\begin{aligned}
\deg\left(zq(z)p'(z)\right) &= 1 + e + (d-1) = d + e, \\
\deg\left(zp(z)q'(z)\right) &= 1 + d + (e-1) = d + e, \\
\deg\left(p(z)q(z)\right) &= d + e.
\end{aligned}$$

Denominator:

$$\begin{aligned}
\deg\left(q(z)p'(z)\right) &= d + e - 1, \\
\deg\left(p(z)q'(z)\right) &= d + e - 1.
\end{aligned}$$

Thus, $\overline{\deg}(R) = e + d$ and $\underline{\deg}(R) = e + d - 1$, provided the coefficients of the numerator's $z^{e+d}$ term and denominator's $z^{e+d-1}$ term are nonzero. If the numerator's $z^{e+d}$ coefficient is nonzero, then it is clear that $\overline{\deg}(R) > \underline{\deg}(R)$. Notice, since each main component in our above list of the numerator and denominator has the same degree, the leading coefficients of the numerator and denominator can easily be calculated using each term's leading coefficient as detailed below:

Numerator:

$$\begin{aligned} \mathrm{lc}\left(zq(z)p'(z)\right) &= d, \\ \mathrm{lc}\left(-zp(z)q'(z)\right) &= e, \\ \mathrm{lc}\left(-p(z)q(z)\right) &= 1. \end{aligned}$$

Denominator:

$$\begin{aligned} \mathrm{lc}\left(p'(z)q(z)\right) &= d, \\ \mathrm{lc}\left(-p(z)q'(z)\right) &= e. \end{aligned}$$

Therefore, $\overline{\mathrm{lc}}(R) = d - e - 1 \neq 0$ if and only if $d \neq e + 1$. In addition we have $\underline{\mathrm{lc}}(R) = d - e$. Thus, $\overline{\deg}(R) > \underline{\deg}(R)$ when $d \neq e + 1$.

Now suppose the coefficient of the numerator's $z^{d+e}$ term is zero, in other words, $d = e + 1$, then $\overline{\deg}(R) \leq d + e - 1$ (with equality depending on the coefficient of the $z^{d+e-1}$ term). Notice, if $d = e + 1$ then $\underline{\deg}(R) = d + e - 1$, since $\underline{\mathrm{lc}}(R) = d - e = (e + 1) - e = 1 \neq 0$. Therefore, if $d = e + 1$ and $\overline{\deg}(R) = d + e - 1$, then $\overline{\deg}(R) = \underline{\deg}(R)$, implying $\infty$ is not a fixed point. On the other hand, if $d = e + 1$ and $\overline{\deg}(R) < d + e - 1$, then $\overline{\deg}(R) < \underline{\deg}(R)$ and $R(\infty) = 0$ so $\infty$ is not a fixed point. Hence, $\infty$ is a fixed point if and only if $d \neq e + 1$.

Recall, to determine the type of fixed point we must evaluate the multiplier

$$\lambda = \left|\frac{1}{R'(\infty)}\right|.$$

If:

- $\lambda > 1$, then $\infty$ is a repelling fixed point;
- $\lambda < 1$, then $\infty$ is an attracting fixed point;
- $\lambda = 1$, then $\infty$ is an indifferent fixed point.

To evaluate the multiplier at $\infty$ we will examine the degrees of both the numerator and denominator of $R'(z)$. To do this we will apply Lemma 6.1.3 to the Newton map, $R(z)$. Note the following relations between the $F(z)$ in Lemma 6.1.3 and the Newton map, $R(z)$:

$$\begin{aligned} f(z) &= zq(z)p'(z) - zp(z)q'(z) - p(z)q(z), \\ g(z) &= q(z)p'(z) - p(z)q'(z), \\ a = \deg(f) &= d + e, \\ b = \deg(g) &= d + e - 1, \\ \mathrm{lc}(f) &= d - e - 1, \\ \mathrm{lc}(g) &= d - e. \end{aligned}$$

Recall, we are assuming $\infty$ is a fixed point, and hence $d \neq e + 1$.

Now, providing $\underline{\mathrm{lc}}(R) \neq 0$, i.e., $d \neq e$, Lemma 6.1.3 states $\overline{\deg}(R') = \underline{\deg}(R')$. Hence, evaluating the derivative at $\infty$ is equivalent to evaluating $\dfrac{\overline{\mathrm{lc}}(R')}{\underline{\mathrm{lc}}(R')}$.

We begin to evaluate $R'(\infty)$ by considering the following:

$$R'(z) - 1 = -\frac{(q(z)p'(z))^2 - (p(z)q'(z))^2 - p(z)q(z)(p''(z)q(z) - p(z)q''(z))}{(q(z)p'(z) - p(z)q'(z))^2}.$$

We first must determine the degree of each term so we can evaluate which contribute to the leading coefficient of each the numerator and denominator. The degree of each term of $R'(z) - 1$ is as follows:

Numerator:

$$\begin{aligned} \deg\left((q(z)p'(z))^2\right) &= 2d + 2e - 2, \\ \deg\left((p(z)q'(z))^2\right) &= 2d + 2e - 2, \\ \deg\left(p(z)q(z)p''(z)q(z)\right) &= 2d + 2e - 2, \\ \deg\left(p(z)q(z)q''(z)p(z)\right) &= 2d + 2e - 2. \end{aligned}$$

Denominator:

$$\begin{aligned} \deg\left(q(z)p'(z)\right) &= e + d - 1, \\ \deg\left(p(z)q'(z)\right) &= d + e - 1. \end{aligned}$$

Therefore, the leading coefficients of the numerator and denominator can be easily calculated by observing the leading coefficients of each term since each is of the same degree. Thus, calculating the leading coefficients of each term we obtain:

Numerator:

$$\begin{aligned} \mathrm{lc}\left((q(z)p'(z))^2\right) &= d^2, \\ \mathrm{lc}\left(-(p(z)q'(z))^2\right) &= -e^2, \\ \mathrm{lc}\left(-p(z)q(z)p''(z)q(z)\right) &= -d(d-1), \\ \mathrm{lc}\left(p(z)q(z)q''(z)p(z)\right) &= e(e-1). \end{aligned}$$

Denominator:

$$\begin{aligned} \mathrm{lc}\{q(z)p'(z)\} &= d, \\ \mathrm{lc}\{-p(z)q'(z)\} &= -e. \end{aligned}$$

Hence, $\overline{\mathrm{lc}}(R'(z) + 1) = d^2 - e^2 - d(d-1) + e(e-1) = d - e$, and $\underline{\mathrm{lc}}(R'(z) + 1) = (d - e)^2$. Providing $d \neq e$, $R'(\infty) = 1 - \dfrac{d - e}{(d - e)^2} = 1 - \dfrac{1}{d - e}$. Therefore, when $d \neq e$ the multiplier at $\infty$ is

$$\lambda = \left|\frac{1}{R'(\infty)}\right| = \left|\frac{1}{1 - \frac{1}{d-e}}\right|.$$

We can deduce:

$$\begin{aligned} \lambda &> 1 \text{ when } d > e, \\ \lambda &< 1 \text{ when } d < e, \text{ and} \\ \lambda &= 1 \text{ never.} \end{aligned}$$

Now suppose $d = e$. Using the notation in Lemma 6.1.3, $b = \deg(g) \leq d + e - 2$, since the coefficient of the $z^{d+e-1}$ term in the denominator is zero (with equality if and only if the $z^{d+e-2}$ term is nonzero). Notice $\deg(f) = a = d + e$. Therefore, since

$$a = d + e > (d + e - 2) + 1 \geq b + 1,$$

Lemma 6.1.3 implies that $\overline{\deg}(R') > \underline{\deg}(R')$.

Thus when $d = e$, $R'(\infty) = \infty$, and the multiplier at $\infty$ is $\left|\frac{1}{R'(\infty)}\right| = 0$. In other words, when $d = e$, $\infty$ is a super-attracting fixed point.

□

As we stated at the end of §2.5 a corollary to Proposition 2.5.4 will be presented now.

**Corollary 3.1.3.** *(to Theorem 2.5.4) Let $R(z)$ be the Newton map of $r(z)$, then $R(z)$ is conjugate to a polynomial if and only if at least one of the following occurs:*

1. *there exists a finite root, $r$ of $r(z)$ such that $R^{-1}(r) = \{r\}$,*
2. *there exists a finite pole, $p$ of $r(z)$ such that $R^{-1}(p) = \{p\}$,*
3. *provided $\overline{\deg}(r) \neq \underline{\deg}(r) + 1$, then $R^{-1}(\infty) = \{\infty\}$.*

*Proof.* By Proposition 2.5.4 we know that $R(z)$ is conjugate to a polynomial if and only if there exists a $w$ such that $R^{-1}(w) = \{w\}$. Notice this $w$ is a fixed point of $R(z)$. By Lemma 6.1.1 we know that the roots and poles of $r(z)$ are the fixed points of $R(z)$. And by Lemma 6.1.4 we know that if $\overline{\deg}(r) \neq \underline{\deg}(r) + 1$ then $\infty$ is a fixed point of $R(z)$.

□

Corollary 3.1.3 is useful as it reduces our search for a $w$ in the entire complex sphere satisfying $R^{-1}(w) = \{w\}$ to just the finite roots and poles of $r(z)$, or, if $\overline{\deg}(r) \neq \underline{\deg}(r) + 1$, then $w = \infty$.

# CHAPTER 7
# ITERATIONS OF THE NEWTON MAP OF SPECIAL CASES

As we mentioned in Chapter 3 we now will explore two special cases. We note the cases we explore in this chapter have been investigated in [4], [6], [9], [10], [11], [13], [24]. We will verify a conjecture made in [13] and extended some of those results. We do so in light of considering these two cases as very special cases in our general theorem which categorizes all degree 2 Newton maps. Moreover, in Chapter 8 we will introduce a new case that has not been previously explored.

## 7.1 The Newton Map of Möbius Transformations

Just as the iteration of Möbius transformations was simple to study, the iteration of the Newton map applied to Möbius transformations is simple as well, and this will be seen through our use of conjugation. However, before we examine the specific location of the basins of attraction and the Julia and Fatou sets, we will consider an example.

Let $r(z) = \dfrac{iz - 3 + i}{2z + 4}$, and consider the Newton map of $r(z)$. In Figure 7.1 the blue region, an open disk, is the basin of attraction of the attracting fixed point $-3i + 1$ (the root of $r(z)$), the red region is the basin of attraction of the attracting fixed point $\infty$, and the circle separating these two basins of attraction is where the repelling fixed point, $-2$, lies (the pole of $r(z)$). In addition, the boundary of the blue region is the Julia set and its complement is the Fatou set.

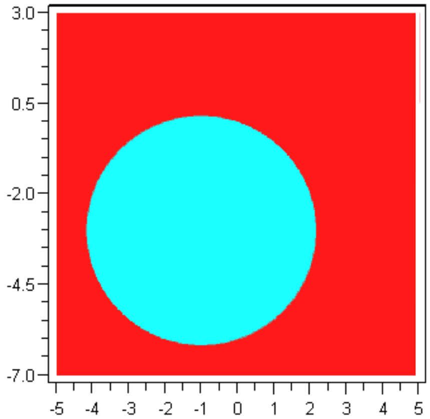


Figure 7.1: Basins of Attraction of the Newton Map of $\dfrac{iz - 3 + i}{2z + 4}$.

By Proposition 2.6.1 we need only consider the Newton map of rational functions with monic polynomials in both the numerator and denominator. However, it is standard to consider Möbius transformations that are not necessarily made up of monic polynomials. Thus, when iterating the Newton map of a Möbius transformation the standard form will be used here, as shown below.

Let $r(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $\alpha\delta - \beta\gamma \neq 0$ and $\alpha \neq 0$, $\gamma \neq 0$. Then

$$R(z) = z - \frac{r(z)}{r'(z)} = \frac{\alpha\gamma z^2 + 2\beta\gamma z + \beta\delta}{\beta\gamma - \alpha\delta}.$$

Notice that $R(z)$ is always a quadratic polynomial.

**Proposition 7.1.1.** *Let $R(z)$ be the Newton map of $r(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, then $R \overset{\overleftarrow{M}}{\sim} S$, where $S(z) = z^2$ and $M(z) = \frac{\alpha\gamma}{\beta\gamma - \alpha\delta} z + \frac{\beta\gamma}{\beta\gamma - \alpha\delta}$.*

*Proof.* The proof is a simple application of Theorem 2.10.1. Using the notation from Theorem 2.10.1 we obtain

$$a = \frac{\alpha\gamma}{\beta\gamma - \alpha\delta}, \qquad b = \frac{\beta\gamma}{\beta\gamma - \alpha\delta}, \qquad d = \frac{\beta\delta}{\beta\gamma - \alpha\delta}$$

and

$$c = \frac{\alpha\gamma}{\beta\gamma - \alpha\delta}\frac{\beta\delta}{\beta\gamma - \alpha\delta} + \frac{\beta\gamma}{\beta\gamma - \alpha\delta} - \left(\frac{\beta\gamma}{\beta\gamma - \alpha\delta}\right)^2 = 0.$$

Hence $S(z) = z^2$ and $M(z) = \dfrac{\alpha\gamma}{\beta\gamma - \alpha\delta} z + \dfrac{\beta\gamma}{\beta\gamma - \alpha\delta}$.

□

J. Corte investigated the Newton map applied to functions of the form $\frac{\alpha z^n + \beta}{\gamma z^n + \delta}$ [13]. She conjectured, for $n = 1$, i.e. for Möbius transformations, the basin of attraction of the root $-\frac{\beta}{\alpha}$ is the disk centered at $-\frac{\beta}{\alpha}$ with radius $\left|\frac{\beta}{\alpha} - \frac{\delta}{\gamma}\right|$. We verify her conjecture and prove further extensions.

**Proposition 7.1.2.** *Let $R(z)$ be the Newton map of $r(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$. Then*

$$\begin{aligned} J(R) &= \left\{ z \,\middle|\, \left|z + \frac{\beta}{\alpha}\right| = \left|\frac{\beta}{\alpha} - \frac{\delta}{\gamma}\right| \right\} \\ F(R) &= \mathbb{C}_\infty \middle\backslash \left\{ z \,\middle|\, \left|z + \frac{\beta}{\alpha}\right| = \left|\frac{\beta}{\alpha} - \frac{\delta}{\gamma}\right| \right\}. \end{aligned}$$

*In addition,*

1. $\left\{z \,\middle|\, \left|z + \frac{\beta}{\alpha}\right| < \left|\frac{\beta}{\alpha} - \frac{\delta}{\gamma}\right|\right\}$ *is the basin of attraction of the fixed point* $-\frac{\beta}{\alpha}$*, and*

2. $\left\{z \,\middle|\, \left|z + \frac{\beta}{\alpha}\right| > \left|\frac{\beta}{\alpha} - \frac{\delta}{\gamma}\right|\right\}$ *is the basin of attraction of the fixed point* $\infty$*.*

*Proof.* By Proposition 7.1.1, $R \sim z^2$, and by Theorem 2.7.7, $M^{-1}(J(z^2)) = J(R)$. In §2.9 we showed that $J(z^2) = S^1 = \left\{z \,\middle|\, |z| = 1\right\}$. Since $M^{-1}(z)$ is a Möbius transformation, $J(R)$ is also a circle, which includes the point $-\frac{\delta}{\gamma}$, given that $M^{-1}(1) = -\frac{\delta}{\gamma}$. By mapping two other points on $S^1$ under $M^{-1}(z)$ it can be seen that $J(R) = \left\{z \,\middle|\, \left|z + \frac{\beta}{\alpha}\right| = \left|\frac{\beta}{\alpha} - \frac{\delta}{\gamma}\right|\right\}$. Therefore, $F(R) = \mathbb{C}_\infty \Big\backslash \left\{z \,\middle|\, \left|z + \frac{\beta}{\alpha}\right| = \left|\frac{\beta}{\alpha} - \frac{\delta}{\gamma}\right|\right\}$. Note that $M^{-1}(z) = \frac{\beta\gamma - \alpha\delta}{\alpha\gamma} z - \frac{\beta}{\alpha} \neq \infty$ for all $z \in S^1$; hence, $J(R)$ will never be a line i.e. a circle going through $\infty$.

By Lemma 6.1.2, since $-\frac{\beta}{\alpha}$ is a root of $r(z)$ it is an attracting fixed point of the Newton map. In addition, Lemma 6.1.2 implies that the pole $-\frac{\delta}{\gamma}$ is a repelling fixed point of the Newton map. Finally, since $\overline{\deg}(r) = \underline{\deg}(r)$, $\infty$ is a super-attracting fixed point, by Lemma 6.1.4.

Next we will find the two basins of attraction of the attracting fixed points $-\frac{\beta}{\alpha}$ and $\infty$. Let $F_0 = \left\{z \,\middle|\, \left|z + \frac{\beta}{\alpha}\right| < \left|\frac{\beta}{\alpha} - \frac{\delta}{\gamma}\right|\right\}$ and $F_1 = \left\{z \,\middle|\, \left|z + \frac{\beta}{\alpha}\right| > \left|\frac{\beta}{\alpha} - \frac{\delta}{\gamma}\right|\right\}$. Notice $-\frac{\beta}{\alpha} \in F_0$ and $\infty \in F_1$.

$F_0$ is the basin of attraction of the attracting fixed point $-\frac{\beta}{\alpha}$ which can be seen by a change of variable. Let $z = w - \frac{\beta}{\alpha}$, then it is easy to see the Newton map becomes $\dfrac{\alpha\gamma}{\beta\gamma - \alpha\delta} w^2 - \dfrac{\beta}{\alpha}$, and by rewriting the basin of attraction as $F_0 = \left\{w \,\middle|\, |w| < \left|\frac{\beta}{\alpha} - \frac{\delta}{\gamma}\right| = \left|\frac{\beta\gamma - \delta\alpha}{\alpha\gamma}\right|\right\}$. It is clear if $w \in F_0$ that $R^n(w) \to 0 - \frac{\beta}{\gamma}$. The same holds true for $F_1$, by using the same change of variable, we see that when $w \in F_1 = \left\{w \,\middle|\, |w| > \left|\frac{\beta}{\alpha} - \frac{\delta}{\gamma}\right|\right\}$, $R^n(w) \to \infty$.

□

A change of variable was used to prove the basins of attraction in Proposition 7.1.2; however, we may also use conjugation to determine the basins of attraction, and this will be done in the proof of Proposition 7.2.2.

## 7.2 The Newton Map of Quadratic Polynomials

In this section we will first give two examples, then prove the claims made in the examples. We will also provide more known general information about quadratic polynomials and the Newton map.

Recall Figure 2.1 exhibited the basins of attraction of the Newton map of a quadratic polynomial having symmetric roots with respect to the imaginary axis.

In Figure 7.2, the basins of attraction of the Newton map applied to $(z-2)(z+5+i)$ are shown. This image is an example of the Newton map of a quadratic polynomial having non-symmetric roots with respect to the imaginary axis. The blue region is the basin of attraction of the root 2 and the green region is the basin of attraction of the root $-5-i$. Note the line separating the two regions is the Julia set, and is the perpendicular bisector of the line segment connecting the two roots. We will see in Proposition 7.2.2 that the Julia set of the Newton map of all quadratic polynomials will be the perpendicular bisector of the line segment connecting the two roots.

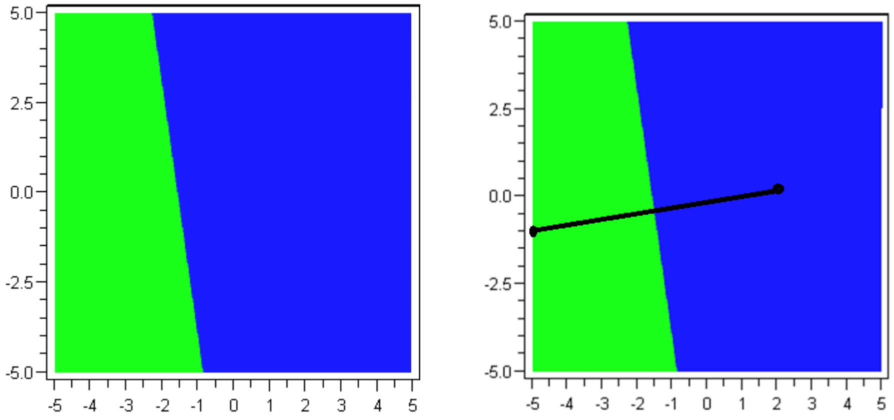


Figure 7.2: Basins of Attraction of the Newton Map of $(z-2)(z+5+i)$ and Connecting Line Segment.

Now, we will provide known information concerning the Newton map applied to quadratic polynomials. Consider a polynomial of the form $r(z) = (z-r_1)(z-r_2)$ for $r_1, r_2 \in \mathbb{C}_\infty$, and $r_1 \neq r_2$, then the Newton map of $r(z)$ is of the form

$$R(z) = \frac{z^2 - r_1 r_2}{2z - (r_1 + r_2)}.$$

Clearly, studying the iteration of this non-trivial rational function would be more complicated than studying the iteration of a conjugated polynomial. Hence, if we can verify $R(z)$ can be conjugated to a polynomial, by Corollary 3.1.3 our study of the iterates of $R(z)$ will be simplified.

For $w \in \{r_1, r_2\}$ where,

$$R^{-1}(z) = z \pm \sqrt{z^2 - (r_1 + r_2)z + r_1 r_2}$$

we see that $R^{-1}(w) = \{w\}$. Therefore, Theorems 2.5.4, 2.5.1 and 2.10.1 tell us that $R(z)$ is conjugate to a polynomial $p(z)$, $\deg(p) = 2$, and $R(z)$ is conjugate to $z^2 + c$, respectively.

Before stating the following proposition about conjugation we will first investigate $R(z)$. Knowing the Newton map converges to roots of the original function, one may hope that if $|z_0 - r_1| < |z_0 - r_2|$, then $R^n(z_0) \to r_1$, and if $|z_0 - r_1| > |z_0 - r_2|$, then $R^n(z_0) \to r_2$. By considering the Möbius transformation $M(z) = \frac{z-r_1}{z-r_2}$, we know if $|M(z)| > 1$, then $|z - r_1| < |z - r_2|$ and if $|M(z)| < 1$, then $|z - r_1| < |z - r_2|$. Recall, the idea of looking at whether $|\cdot| > 1$ or $|\cdot| < 1$ was used in §2.9 when investigating the iteration of $f(z) = z^2$. $M(z)$ is an educated guess of the Möbius transformation that will conjugate $R(z)$ to a polynomial. The following well-known proposition shows our educated guess is indeed the correct one. The following two propositions are credited to Cayley, see [10] and [11].

**Proposition 7.2.1.** *Let $R(z)$ be the Newton map of the quadratic polynomial $r(z) = (z - r_1)(z - r_2)$. Then $R \overset{\overleftarrow{M}}{\sim} S$ where $S(z) = z^2$, and $M(z) = \dfrac{z - r_1}{z - r_2}$.*

*Proof.* It is a straightforward calculation to show that $MRM^{-1}(z) = z^2$ where $R(z) = \dfrac{z^2 - r_1 r_2}{2z - (r_1 + r_2)}$ and $M(z) = \dfrac{z - r_1}{z - r_2}$. Note, $M(z)$ is a Möbius transformation, since $r_1 \neq r_2$.

□

**Proposition 7.2.2.** *Let $R(z)$ be the Newton map of $r(z) = (z - r_1)(z - r_2)$ then $J(R)$ is the perpendicular bisector of the line segment connecting the two distinct roots $r_1$ and $r_2$. Specifically,*

$$J(R) = \begin{cases} \left\{ z = x + iy \;\middle|\; y = \dfrac{(r_2 - r_1)_x}{(r_1 - r_2)_y} x + \dfrac{|r_1|^2 - |r_2|^2}{2(r_1 - r_2)_y} \right\} & \text{if} \quad (r_1)_y \neq (r_2)_y \\ \left\{ z = x + iy \;\middle|\; x = \dfrac{(r_1)_x + (r_2)_x}{2} \right\} & \text{if} \quad (r_1)_y = (r_2)_y, \end{cases}$$

*where $z_x$ indicates the real part of $z$ and $z_y$ indicates the imaginary part of $z$. In addition,*

1. *$\left\{ z \;\middle|\; |z - r_1| < |z - r_2| \right\}$ is the basin of attraction of $r_1$, and*
2. *$\left\{ z \;\middle|\; |z - r_1| > |z - r_2| \right\}$ is the basin of attraction of $r_2$.*

We provide a proof for completeness.

*Proof.* By Theorem 2.7.7 and §2.9, $M^{-1}(J(z^2)) = M^{-1}(S^1) = J(R)$. Thus, $J(R)$ is a circle, since $M^{-1}(z) = \dfrac{r_1 - r_2 z}{1 - z}$ is a Möbius transformation. Consider the two points on the unit circle $z = \pm 1$. Mapping these under $M^{-1}(z)$ we see that: $M^{-1}(1) = \infty$ and $M^{-1}(-1) = \dfrac{r_1 + r_2}{2}$. Therefore, $J(R)$ is a line going through the midpoint of the line segment connecting the two roots $r_1$ and $r_2$.

We can calculate the equation of the line by considering one more point on the unit circle

$$M^{-1}(i) = \frac{ir_2 - r_1}{i - 1} = \frac{r_2(1 - i) + r_1(1 + i)}{2}.$$

Calculating the line between the points $\dfrac{r_1 + r_2}{2}$ and $\dfrac{r_1 + r_2}{2} + i\dfrac{r_1 - r_2}{2}$ and supposing $(r_1)_y \neq (r_2)_y$ we obtain $y = \dfrac{(r_2 - r_1)_x}{(r_1 - r_2)_y}x + \dfrac{|r_1|^2 - |r_2|^2}{2(r_1 - r_2)_y}$. Hence,

$$J(R) = \left\{ z = x + iy \;\middle|\; y = \frac{(r_2 - r_1)_x}{(r_1 - r_2)_y}x + \frac{|r_1|^2 - |r_2|^2}{2(r_1 - r_2)_y} \right\}.$$

Now, suppose $(r_1)_y = (r_2)_y$. Calculating the equation of the line passing through the points $\dfrac{r_1 + r_2}{2}$ and $\dfrac{r_1 + r_2}{2} + i\dfrac{r_1 - r_2}{2}$, we obtain $x = \dfrac{(r_1)_x + (r_2)_x}{2}$. Therefore,

$$J(R) = \left\{ z = x + iy \;\middle|\; x = \frac{(r_1)_x + (r_2)_x}{2} \right\}.$$

Rather than relying on the calculations above to show the lines are perpendicular, we can use a geometric property of Möbius transformations. Recall, Möbius transformations preserve angles. Since $J(z^2) = S^1$ is perpendicular to the line connecting the two attracting fixed points of $z^2$, $\{0, \infty\}$. We know that $M^{-1}(S^1)$ will also be perpendicular to the line connecting the two attracting fixed points of $R(z)$, $\{r_1, r_2\} = M^{-1}\{0, \infty\}$.

Next we want to show that $\left\{ z \;\middle|\; |z - r_1| < |z - r_2| \right\}$ is the basin of attraction of $r_1$. We know that $R \overset{\overleftarrow{M}}{\sim} S$ and $MR^n M^{-1}(z) = S^n(z)$, where $S(z) = z^2$ and we have also proved that $\left\{ z \;\middle|\; |z| < 1 \right\}$ is the basin of attraction of 0 for $S(z)$. For $|z| < 1$, $S^n(z) \to 0$. Proposition 2.5.3 implies $MR^n M^{-1}(z) \to 0$ as well. Hence, for $|M(z)| < 1$, in other words, $|z - r_1| < |z - r_2|$, $S^n(M(z)) \to 0$, and since $S^n(M(z)) = M(R^n(z))$ this implies that $MR^n(z) \to 0$. Notice $M^{-1}(0) = r_1$, so by applying $M^{-1}$ to both sides of the above convergence we see that $M^{-1}(M(R(z))) \to M^{-1}(0) = r_1$ for all $z$ such that $|z - r_1| < |z - r_2|$.

Therefore, $\left\{z \;\middle|\; |z - r_1| < |z - r_2|\right\}$ is the basin of attraction of the attracting fixed point $r_1$.

The same method as above can be applied to determine the basin of attraction of $r_2$ is $\left\{z \;\middle|\; |z - r_1| > |z - r_2|\right\}$.

□

# CHAPTER 8
# THE NEWTON MAP OF $\frac{(Z-A)(Z-B)}{Z-C}$

In Chapter 7 we considered the Newton map of the rational function $\frac{\alpha z+\beta}{\gamma z+\delta}$, a Möbius transformation, and also explored the Newton map of a quadratic polynomial, both of which have been investigated previously. In this chapter we will consider the Newton map of the simplest, nontrivial, quadratic rational function, $\frac{(z-r_1)(z-r_2)}{z-p}$ which has not been covered in the literature. This rational function was a specific case of the main theorem of Chapter 3; however, for completeness we will investigate this in detail here.

Figure 8.1 shows the two basins of attraction of the Newton map of the rational function $r(z) = \dfrac{z^2-4}{z+3-i}$. Notice the green region is the basin of attraction of the root 2, and the blue region, an open disk, is the basin of attraction of the root $-2$. In addition, the circle separating these two regions contains the pole $-3+i$, and is in fact the Julia set.

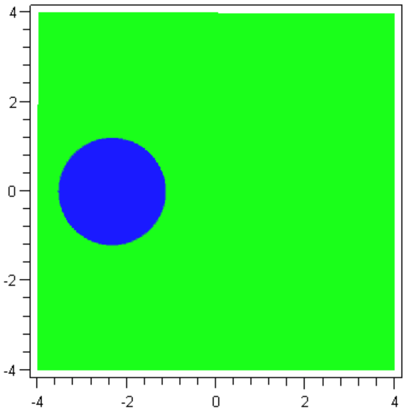


Figure 8.1: Basins of Attraction of the Newton Map of $\dfrac{z^2-4}{z+3-i}$.

Recall when iterating the Newton map of a rational function it may be assumed the rational function is a quotient of two coprime monic polynomials (Proposition 2.6.1). Therefore, instead of considering the quotient of any quadratic function and any linear function we instead will consider

$$r(z) = \frac{(z-r_1)(z-r_2)}{z-p} \quad \text{where} \quad r_1, r_2, p \in \mathbb{C} \quad \text{and} \quad r_1 \neq r_2, r_1 \neq p \quad \text{and} \quad r_2 \neq p. \tag{8.1}$$

The Newton map of $r(z)$ is given by:

$$R(z) = z - \frac{(z-r_1)(z-r_2)(z-p)}{z^2 - 2zp + r_1p + r_2p - r_1r_2} = \frac{z^2(r_1 + r_2 - p) - 2zr_1r_2 + r_1r_2p}{z^2 - 2zp + r_1p + r_2p - r_1r_2}. \tag{8.2}$$

The same process used in §7.2, of conjugating the Newton map of a quadratic polynomial to $z^2$, can be used here to find the conjugated polynomial of the Newton map applied to a rational function of the form (8.1).

If $R(z)$ is conjugate to a polynomial, by Corollary 3.1.3 we need only look at the finite roots or poles of $r(z)$, since $\overline{\deg}(r) = \underline{\deg}(r) + 1$, $\infty$ is not a fixed point of $R(z)$. A quick calculation shows the inverse of (8.2) is

$$R^{-1}(z) = \frac{(r_1r_2 - pz) \pm \sqrt{(z-r_1)(z-r_2)(r_1-p)(r_2-p)}}{r_1 + r_2 - p - z}.$$

If $z = r_1$, then

$$R^{-1}(r_1) = \left\{\frac{r_1r_2 - pr_1}{r_1 + r_2 - p - r_1}\right\} = \left\{\frac{r_1(r_2 - p)}{r_2 - p}\right\} = \{r_1\}.$$

And if $z = r_2$, then

$$R^{-1}(r_2) = \left\{\frac{r_1r_2 - r_2p}{r_1 + r_2 - p - r_1}\right\} = \left\{\frac{r_2(r_2 - p)}{r_2 - p}\right\} = \{r_2\}.$$

Finally if $z = p$, then

$$R^{-1}(p) = \left\{\frac{r_1r_2 - p^2 \pm \sqrt{(p-r_1)^2(p-r_2)^2}}{r_1 + r_2 - 2p}\right\} \neq \{p\},$$

since $r_1 \neq p$ and $r_2 \neq p$.

Therefore, since there exists a point, $w \in \mathbb{C}_\infty$, (actually two) such that $R^{-1}(w) = \{w\}$, $R(z)$ is conjugate to a quadratic polynomial by Corollary 3.1.3. Note this work is a simplified version of Theorem 3.1.4, Case ($2b$).

**Proposition 8.1.1.** *Let* $R(z)$ *be the Newton map of* $r(z) = \dfrac{(z-r_1)(z-r_2)}{z-p}$, *then* $R \overset{\overleftarrow{M}}{\sim} S$ *when*

$$M(z) = \frac{(z-r_1)(r_2-p)}{(z-r_2)(r_1-p)} \qquad \text{and} \qquad S(z) = z^2.$$

*Proof.* Simplifying $MRM^{-1}(z)$ one obtains $S(z)$. However, we must verify that $M(z)$ is indeed a Möbius transformation. For $M(z)$ to be a Möbius transformation

$$r_1(r_2 - p)(r_1 - p) - r_2(r_2 - p)(r_1 - p) \neq 0,$$

or equivalently, $(r_2 - p)(r_1 - p)(r_1 - r_2) \neq 0$. The result follows since the roots and poles are all distinct.

$\square$

Recall that $z_x$ indicates the real part of $z$ and $z_y$ indicates the imaginary part of $z$.

**Proposition 8.1.2.** *Let $R(z)$ be the Newton map of $r(z)$, which is of the form (8.1). Then the Julia set of $R(z)$ is a circle passing through $\infty$ when the pole is equidistant from the two roots, and a circle not passing through the point $\infty$ when the pole is not equidistant to the two roots. More specifically,*

$$J(R) = \begin{cases} \left\{ re^{i\theta} + C \;\middle|\; \theta \in [0, 2\pi) \right\}, & \text{if } |p - r_1| \neq |p - r_2|, \\ \left\{ z = x + iy \;\middle|\; y = \dfrac{(r_1 - r_2)_x}{(r_2 - r_1)_y} x + p_y - p_x \dfrac{(r_1 - r_2)_x}{(r_2 - r_1)_y} \right\}, & \\ & \text{if } |p - r_1| = |p - r_2| \text{ and } (r_1)_y \neq (r_2)_y, \\ \left\{ z = x + iy \;\middle|\; x = p_x \right\}, & \text{if } |p - r_1| = |p - r_2| \text{ and } (r_1)_y = (r_2)_y, \end{cases}$$

*where the radius, r, and the center of the circle, C in the first case, are given by,*

$$r = \frac{|(r_1 - r_2)(r_1 - p)(r_2 - p)|}{\left|(r_1 - r_2)_x (r_1 + r_2 - 2p)_x + (r_1 - r_2)_y (r_1 + r_2 - 2p)_y\right|},$$

$$C = \frac{(r_1 - r_2)_x((r_1)_x(r_2)_x - p_x^2 - p_y^2) + (r_1)_y(r_2)_x((r_1)_y - 2p_y) - (r_1)_x(r_2)_y((r_1)_x - 2p_y)}{(r_1 - r_2)_x(r_1 + r_2 - 2p)_x + (r_1 - r_2)_y(r_1 + r_2 - 2p)_y} +$$
$$i\frac{(r_1 - r_2)_y((r_2)_y(r_1)_y - p_x^2 - p_y^2) - (r_1)_y(r_2)_x((r_1)_y - 2p_x) + (r_1)_x(r_2)_y((r_1)_x - 2p_x)}{(r_1 - r_2)_x(r_1 + r_2 - 2p)_x + (r_1 - r_2)_y(r_1 + r_2 - 2p)_y}.$$

*Proof.* By Proposition 8.1.1 we know that $R \overset{\overleftarrow{M}}{\sim} S$, where

$$M(z) = \frac{(z - r_1)(r_2 - p)}{(z - r_2)(r_1 - p)} \text{ and } S(z) = z^2.$$

Using the fact that $J(S) = S^1$ and Theorem 2.7.7, we see that

$$J(R) = M^{-1}(J(S)) = M^{-1}(S^1),$$

thus $J(R)$ is a circle. A simple calculation yields the Möbius transformation that will transform the unit circle to the Julia set of $R(z)$,

$$M^{-1}(z) = \frac{zr_2(r_1 - p) + r_1(p - r_2)}{z(r_1 - p) + (p - r_2)}.$$

First suppose that $M^{-1}(J(S))$ is a circle passing through $\infty$. This is true if and only if $M^{-1}(z) = \infty$ for some $z \in S^1$. Notice

$$\frac{-r_2 z(r_1 - p) + r_1(r_2 - p)}{-z(r_1 - p) + (r_2 - p)} = \infty$$

if and only if $-z(r_1 - p) + (r_2 - p) = 0$ if and only if $z = \frac{r_2 - p}{r_1 - p}$. Since $z \in S^1$ we have $|z| = 1$ which implies that $\left|\frac{r_2 - p}{r_1 - p}\right| = 1$, rewritten as $|r_2 - p| = |r_1 - p|$. In other words, if $p$ is equidistant from $r_1$ and $r_2$, then $J(R)$ is a line.

Finding two points on this line will give us the equation of the Julia set. Note for $p$ (equidistant or not to the roots $r_1$ and $r_2$): $M^{-1}(1) = p$. Therefore, whether or not $p$ is equidistant from $r_1$ and $r_2$, we know that $J(R) = M^{-1}(S^1)$ is a line passing through $p$. Since $M$ is a Möbius transformation and $S^1$ is perpendicular to the line connecting 0 and $\infty$, the two attracting fixed points of $z^2$, we know that $J(R)$ is the perpendicular bisector of the line segment connecting $r_1$ and $r_2$, the two attracting fixed points of $R(z)$. There is a more elementary proof to show that $J(R)$ is the perpendicular bisector, and we will do so after this proof.

Previously, we considered $p$ equidistant from $r_1$ and $r_2$. Now, suppose that $|p - r_1| \neq |p - r_2|$. This implies $J(R)$ is a circle, not going through the point at $\infty$, since $M^{-1}(z) = \infty$ if and only if $|p - r_1| = |p - r_2|$. Therefore, to find the exact equation of the circle we will consider the images of three points: $1, i$ and $-1$. Let,

$$\begin{aligned}
A &= M^{-1}(1) &&= p,\\
B &= M^{-1}(i) &&= \frac{r_1(p - r_2) + ir_2(r_1 - p)}{(p - r_2) + i(r_1 - p)},\\
D &= M^{-1}(-1) &&= \frac{r_1(r_2 - p) + r_2(r_1 - p)}{(r_1 - p) + (r_2 - p)},
\end{aligned}$$

and denote the center of the circle as $C = h + ik$. By solving the following two equations:

$$|A - C| = |B - C| \qquad \text{and} \qquad |A - C| = |D - C|$$

for $h$ and $k$, we obtain the real and imaginary parts of the center of the Julia set,

$$\begin{aligned} h &= \frac{(r_1 - r_2)_x((r_1)_x(r_2)_x - p_x^2 - p_y^2) + (r_1)_y(r_2)_x((r_1)_y - 2p_y) - (r_1)_x(r_2)_y((r_1)_x - 2p_y)}{(r_1 - r_2)_x(r_1 + r_2 - 2p)_x + (r_1 - r_2)_y(r_1 + r_2 - 2p)_y}, \text{ and} \\ k &= \frac{(r_1 - r_2)_y((r_2)_y(r_1)_y - p_x^2 - p_y^2) - (r_1)_y(r_2)_x((r_1)_y - 2p_x) + (r_1)_x(r_2)_y((r_1)_x - 2p_x)}{(r_1 - r_2)_x(r_1 + r_2 - 2p)_x + (r_1 - r_2)_y(r_1 + r_2 - 2p)_y}. \end{aligned}$$

Substituting the above values of $h$ and $k$ and solving $|A - C| = r$ for $r$, we obtain the radius of the Julia set:

$$r = \frac{\sqrt{\left[(r_1 - r_2)_x^2 + (r_1 - r_2)_y^2\right]\left[(r_1 - p)_x^2 + (r_1 - p)_y^2\right]\left[(r_2 - p)_x^2 + (r_2 - p)_y^2\right]}}{\left|(r_1 - r_2)_x(r_1 + r_2 - 2p)_x + (r_1 - r_2)_y(r_1 + r_2 - 2p)_y\right|}.$$

Therefore, if $p$ does not satisfy $|r_1 - p| = |p - r_2|$ we have $J(R)$ as a circle with the desired form.

$\square$

In the proof above we used the properties of Möbius transforamtions. We can also use a more elementary approach to showing $J(R)$ is the perpendicular bisector is by showing another point on the line is also equidistant from $r_1$ and $r_2$. We will prove this by considering the point $D = M^{-1}(-1)$. Evaluating the distance between the point $D$, which lies on the Julia set, and $r_1$ we can compute the following:

$$\begin{aligned} |D - r_1| &= \left|\frac{r_2(r_1 - p) + r_1(r_2 - p)}{(r_1 - p) + (r_2 - p)} - r_1\right| \\ &= \left|\frac{r_2(r_1 - p) + r_1(r_2 - p) - r_1(r_1 - p) - r_1(r_2 - p)}{(r_1 - p) + (r_2 - p)}\right| \\ &= \left|\frac{r_2(r_1 - p) - r_1(r_1 - p)}{(r_1 - p) + (r_2 - p)}\right| \\ &= \left|\frac{(r_2 - r_1)(r_1 - p)}{(r_1 - p) + (r_2 - p)}\right|. \end{aligned}$$

Now consider the distance from $D$ and $r_2$, and recall we are assuming $|r_1 - p| = |r_2 - p|$:

$$
\begin{aligned}
|D - r_2| &= \left|\frac{r_2(r_1 - p) + r_1(r_2 - p)}{(r_1 - p) + (r_2 - p)} - r_2\right| \\
&= \left|\frac{r_2(r_1 - p) + r_1(r_2 - p) - r_2(r_1 - p) - r_2(r_2 - p)}{(r_1 - p) + (r_2 - p)}\right| \\
&= \left|\frac{r_1(r_2 - p) - r_2(r_2 - p)}{(r_1 - p) + (r_2 - p)}\right| \\
&= \left|\frac{(r_1 - r_2)(r_2 - p)}{(r_1 - p) + (r_2 - p)}\right| \\
&= \left|\frac{(r_1 - r_2)(r_1 - p)}{(r_1 - p) + (r_2 - p)}\right| \\
&= |D - r_1|.
\end{aligned}
$$

Therefore, if $p$ satisfies $|r_1 - p| = |r_2 - p|$ the Julia set of the Newton map is the perpendicular bisector of the line segment connecting $r_1$ and $r_2$. This line can be written in the forms as stated in the theorem, for $(r_1)_y = (r_2)_y$ and for $(r_1)_y \neq (r_2)_y$.

Notice the similarity of the Julia set being the perpendicular bisector for the different original functions stated in Proposition 7.2.2 and in Proposition 8.1.2.

**Corollary 8.1.3.** *Let $R(z)$ be the Newton map of the reduced rational function $r(z) = \dfrac{(z - r_1)(z - r_2)}{z - p}$ where $r_1, r_2, p \in \mathbb{R}$, then*

$$
J(R) = \begin{cases}
\left\{ \left|\dfrac{(r_1 - p)(r_2 - p)}{(r_1 - p) + (r_2 - p)}\right| e^{i\theta} + \dfrac{r_1 r_2 - p^2}{(r_1 - p) + (r_2 - p)} \,\middle|\, \theta \in [0, 2\pi) \right\} & \textit{if } |p - r_1| \neq |p - r_2|, \\
\{z = x + iy \,|\, x = p,\ y \in \mathbb{R}\} & \textit{otherwise.}
\end{cases}
$$

*Proof.* Letting $(r_1)_y, (r_2)_y$ and $p_y$ equal 0 the result follows from Proposition 8.1.2.

□

**Proposition 8.1.4.** *Let $r(z)$ be the function as stated in equation (8.1). If $p$ satisfies $|p - r_1| = |p - r_2|$ then*

1. $\{z \,\big|\, |z - r_1| < |z - r_2|\}$ *is the basin of attraction of* $r_1$,
2. $\{z \,\big|\, |z - r_1| > |z - r_2|\}$ *is the basin of attraction of* $r_2$.

*If $p$ does not satisfy $|p - r_1| = |p - r_2|$ but does satisfy $|r_i - p| < |r_j - p|$ for $i \neq j$, then*

1. $\{z \big| |z-C| < r\}$ *is the basin of attraction of* $r_i$*,*
2. $\{z \big| |z-C| > r\}$ *is the basin of attraction of* $r_j$*.*

*where r and C are defined in Proposition 8.1.2*

*Proof.* The proof is similar to the proof of Proposition 7.2.2. □

# CHAPTER 9
# TWO ITERATES OF THE FUNCTION $F(Z) = \frac{AZ^2+BZ+C}{DZ^2+EZ+F}$

Finally, we have investigated the first two iterations of an arbitrary degree 2 rational function. As discussed in §2.8 the iteration of Möbius transformations, or degree 1 rational functions, is as simple as matrix multiplication. However, this chapter shows that as soon as one goes beyond degree 1 rational functions the iterations quickly become complicated.

One reason for the complication is the property of the degree of composition of functions. Recall, the degree of the $n$th iteration of $r(z)$ is given by $\deg(r^n) = [\deg(r)]^n$. Therefore, since Möbius transformations are of degree 1, the degree of the $n$th iteration does not increase. However, the degree of the $n$th iteration of a rational map of degree 2 increases to $2^n$.

In addition, while there are at most 6 coefficients in the initial degree 2 rational function, by the third iteration there are at most 18 coefficients. This also explains why simple matrix multiplication cannot be applied to the iteration of rational maps of degree greater than one, since the number of coefficients may increase.

Let

$$f(z) = \frac{a_{2,\alpha}z^2 + a_{1,\alpha}z + a_{0,\alpha}}{a_{2,\beta}z^2 + a_{1,\beta}z + a_{0,\beta}}$$

be an arbitrary degree 2 rational function.

For notations sake, let $\sigma = \alpha$ for the numerator, and $\sigma = \beta$ for the denominator, where $\delta_{n,m}$ is the Kronecker delta function and define

$$p_\sigma(n,m) = \begin{cases} a_{2,\sigma} & \text{if } n = m = \alpha \\ a_{1,\sigma} & \text{if } n \neq m \\ a_{0,\sigma} & \text{if } n = m = \beta. \end{cases}$$

Then the $z^k$ coefficient of $f^2(z)$ is:

$$c_{k,\sigma} = \sum_{i+j=k} \Big(2^{1-\delta_{i,j}} p_\sigma(\alpha,\alpha) a_{i,\alpha} a_{j,\alpha} + 2^{-\delta_{i,j}} p_\sigma(\alpha,\beta) a_{i,\alpha} a_{j,\beta}$$

$$+2^{-\delta_{i,j}} p_\sigma(\beta,\alpha) a_{i,\beta} a_{j,\alpha} + 2^{1-\delta_{i,j}} p_\sigma(\beta,\beta) a_{i,\beta} a_{j,\beta}\Big).$$

So,

$$f^2(z) = \frac{\sum_{k=0}^{2^2} c_{k,\alpha} z^k}{\sum_{k=0}^{2^2} c_{k,\beta} z^k}.$$

The 3rd iteration, where $c_{k,\sigma}$ is the coefficient of the $z^k$ term in the 2nd iteration, and when $\sigma = \alpha$ this indicates the numerator's coefficient, and $\sigma = \beta$ indicates the denominator's coefficient,

$$\sum_{n=0}^{2^3} z^n \sum_{k=0}^{4} (k)!(4-k)!c_{k,\sigma} \sum_{\substack{i_0+i_1+i_2=k \\ j_0+j_1+j_2=4-k \\ \sum_{\ell,m=0}^{2} \ell i_\ell + m j_m = n}} \frac{1}{i_0!i_1!i_2!j_0!j_1!j_2!} (a_{0,\alpha})^{i_0}(a_{1,\alpha})^{i_1}(a_{2,\alpha})^{i_2}(a_{0,\beta})^{j_0}(a_{1,\beta})^{j_1}(a_{2,\beta})^{j_2}$$

$$= \sum_{n=0}^{2^3} z^n \sum_{k=0}^{4} (k)!(4-k)!c_{k,\sigma} \sum_{\substack{\sum_{\ell=0}^{2} i_\ell = k \\ \sum_{m=0}^{2} j_m = 4-k \\ \sum_{\ell,m=0}^{2} \ell i_\ell + m j_m = n}} \prod_{\ell,m=0}^{2} \frac{1}{i_\ell ! j_m !} (a_{\ell,\alpha})^{i_\ell} (a_{m,\beta})^{j_m}.$$

For example, the coefficient of $z^7$ is

$$c_{0,\sigma} 4 a_{2,\beta}^3 a_{1,\beta} + c_{1,\sigma}\left(3 a_{2,\alpha} a_{2,\beta}^2 a_{1,\beta} + a_{1,\alpha} a_{2,\beta}^3\right) + c_{2,\sigma}\left(2 a_{2,\alpha} a_{1,\alpha} a_{2,\beta}^2 + 2 a_{2,\alpha}^2 a_{2,\beta} a_{1,\beta}\right)$$

$$+ c_{3,\sigma}\left(3 a_{2,\alpha}^2 a_{1,\alpha} a_{2,\beta} + a_{2,\alpha}^3 a_{1,\beta}\right) + c_{4,\sigma} 4 a_{2,\alpha}^3 a_{1,\alpha}.$$

Notice, when $\sigma = \alpha$ this is the coefficient of $z^7$ of the numerator and when $\sigma = \beta$ this is the coefficient of $z^7$ of the denominator.

Notice these formulas hold for any degree 2 rational function, not just for ones in the Newton map form. However, any degree 2 Newton map is of the form $f(z)$ with more specific coefficients. For example, the Newton map of $r(z) = \dfrac{(z-a)(z-b)}{(z-c)}$ is

$$R(z) = \frac{(a+b-c)z^2 - 2zab + abc}{z^2 - 2zc + (ac + bc - ab)}.$$

# CHAPTER 10
# CONCLUSION

We have extended two properties of the Newton map of polynomials to the Newton map of rational functions. The property of the degree of the Newton map and the property of the fixed points and type of fixed points throughout the complex sphere were stated in terms of the original rational function. We have also added onto a well-known result about rational functions being conjugate to polynomials to look at our specific rational function, Newton map, and stated exactly when this is conjugate to a polynomial.

We were able to categorize of all Newton maps that are conjugate to $z^2 + c$. This categorization is useful for what is known about the iteration of quadratic polynomials and their connection to the Mandelbrot set. This proof used two lemmas that we stated above: the degree of the Newton map, and when the Newton map is conjugate to a polynomial. The proof also used well-known results of iteration theory. This categorization implies that if the Newton map is conjugate to $z^2 + c$ then the Julia set is connected.

We did not use the same techniques to categorize all Newton maps that are conjugate to a cubic polynomial because nice simplifications did not occur when calculating $R^{-1}(w)$ for $w$ a root or pole of $r(z)$. However, we were able to state that no Newton maps of rational functions are conjugate to $z^3$ by comparing the number of types of fixed points of all degree 3 Newton maps and $z^3$.

Finally, we have written explicitly the first two iterations of a degree two rational function.

There are a few natural extensions on this work that can be considered.

First, we only categorized degree 2 Newton maps that are conjugate to polynomials. It would be interesting to categorize *all* degree 2 Newton maps. The following 6 rational functions do have Newton maps of degree 2 and are not conjugate to polynomials. Notice the first four functions have similar forms to those stated in Theorem 3.1.4, with different restrictions on the powers. Functions 2 and 3 are of the same form; however, there are two different restrictions on the multiplicities, so are stated twice.

1. $r(z) = (z - r_1)^{d_1}(z - r_1)^{d_2}$ where $d_1, d_2 > 1$,
2. $r(z) = \dfrac{(z - r_1)^{d_1}}{(z - p_1)^{e_1}}$ where $d_1 \neq e_1$ and $d_1 \neq e_1 + 1$,
3. $r(z) = \dfrac{(z - r_1)^{d_1}}{(z - p_1)^{e_1}}$ where $d_1 > 1$ and $d_1 \neq e_1 + 1$,

4. $r(z) = \dfrac{(z - r_1)^{d_1}(z - r_2)^{d_2}}{(z - p_1)^{d_1+d_2-1}}$ where $d_1, d_2 > 1$,
5. $r(z) = \dfrac{1}{(z - p_1)^{e_1}(z - p_2)^{e_2}}$,
6. $r(z) = \dfrac{(z - r_1)^{e_1+e_1+1}}{(z - p_1)^{e_1}(z - p_2)^{e_2}}$.

It is also interesting to note that the Julia set of all Newton maps of polynomials is connected, as we have mentioned previously in Chapter 3. However, as Figure 10.1 would suggest the Newton map of rational functions is not always connected. The white indicates the points that do not converge to ∞, the Julia set, and the shades of blue indicate the points that do converge to ∞, the Fatou set, (darker blue indicates faster convergence) of the Newton map of $r(z) = \dfrac{1}{(z-1)(z+i)}$. Notice this is function 5 in the above list of functions whose degree 2 Newton map is not conjugate to a polynomial.

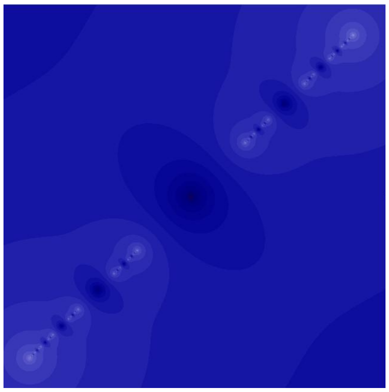
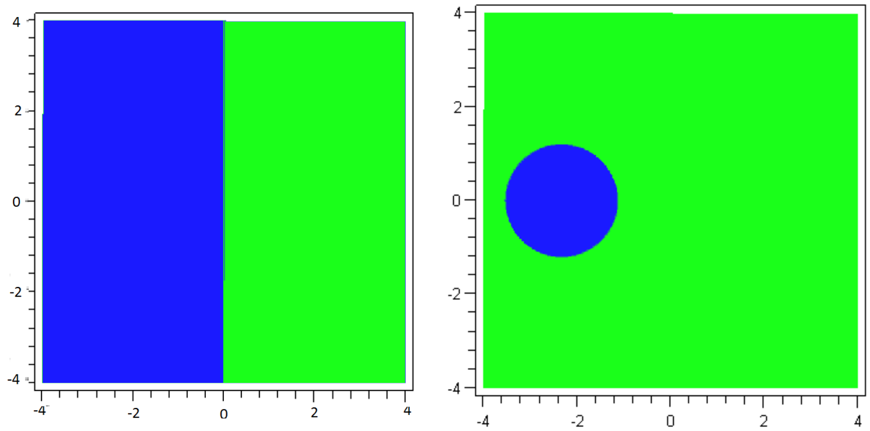


Figure 10.1: Julia set of Newton map that is totally disconnected.

A few natural questions arise. Can one explicitly state the Julia set of the Newton map of what rational functions is connected? not connected? We do know that if the Newton map has only one repelling or indifferent fixed point then it is connected [39]. So one may reduce the list of rational functions to consider by looking at repelling fixed points, since the Newton map of rational functions have no indifferent fixed points.

We have shown the Newton map of rational functions is never conjugate to $z^3$. What about conjugation to other cubic polynomials? And then can this be extended to categorizing all degree 3 Newton maps?

Finally, it is interesting to note that the basins of attraction for the Newton map of $z^2 - 4$ and $\dfrac{z^2-4}{z+3-i}$ have some type of containment, see Figure 10.2. Is there a containment of all basins of attraction for the Newton map of $p(z)$ and the Newton map of $\dfrac{p(z)}{z - p_1}$ for $p_1 \in \mathbb{C}$?



(a) Basins of Attraction for the Newton Map of $z^2 - 4$

(b) Basins of Attraction for the Newton Map of $\frac{z^2-4}{z+3-i}$

Figure 10.2: Containment of Basins of Attraction.

# BIBLIOGRAPHY

[1] Amat, S., Busquier, S., and Plaza, S., *Review of some iterative root-finding methods from a dynamical point of view*, Sci. Ser. A Math. Sci. (N.S.), **10** (2004), no. 10, 3-35, MR2127479 (2006b:37081)

[2] Amat, S., Busquier, S., Plaza, S., *Dynamics of a family of third-order iterative methods that do not require using second derivatives*, Appl. Math. Comput. **154** (2004) no. 3, 735–746, MR2072817

[3] Baker, I.N., *Fixpoints of polynomials and rational functions*, J. London Math. Soc., **39** (1964), no. 1, 615-622, MR0169989 (30 #230)

[4] Beardon, A.F., *Iteration of rational functions*, Springer-Verlag, New York, 2000.

[5] Blanchard, P. *Complex analytic dynamics on the Riemann sphere*, Bull. Amer. Math. Soc. (N.S.) **11** 1984, no. 1, 85–141, MR0741725 (85h:58001)

[6] Blanchard, P. *The dynamics of Newton's method*, (Complex dynamical systems, Cincinnati, OH, 1994), Proc. Sympos. Appl. Math., 49, Amer. Math. Soc., Providence, RI, 1994, 139–154, MR1315536

[7] Bridger, M., *Looking at the Mandelbrot set*, College Math. J. **19** (1988), no. 4, 353-363, MR0962486

[8] Brolin, H., *Invariant sets under iteration of rational functions*, Ark. Mat. **6** (1965), no. 2, 103-144, MR0194595 (33 #2805)

[9] Cayley, M.A., *Application of the Newton-Fourier method to the imaginary root of an equation*, Quart. J. Math., **16** (1879), 179-185

[10] Cayley, M.A., *The Newton-Fourier imaginary problem*, Amer. J. Math., **2** (1879), 97

[11] Cayley, M.A., *On the Newton-Fourier imaginary problem.* Proc. Cambridge Phil. Soc. **3** (1980), 231-232

[12] Conway, J.B, *Functions of one complex variable I*, Springer, New York, 1978

[13] Corte, J. *Fractal images generated by Newton's method*, Thesis, University of Tennessee, Knoxville, 2003.

[14] Curry, J. H., Garnett, L., Sullivan, D., *On the iteration of a rational function: Computer experiments with Newton's method* Commun. Math. Phys. **91** (1983), no 2, 267–277, MR0723551 (85e:30040)

[15] Devaney, R.L., *An Introduction to chaotic dynamical systems*, Westview Press, Colorado, 2003

[16] Devaney, R. L., Unveiling the Mandelbrot Set. Online in Plus Magazine, 2006.

[17] Devaney, R. L., *The complex dynamics of quadratic polynomials*. (Complex dynamical systems, Cincinnati, OH, 1994), Proc. Sympos. Appl. Math., 49, Amer. Math. Soc., Providence, RI, 1994. 1–29, MR1315532

[18] Douady, A. and Hubbard, J., *Etudes dynamique des polynômes complexes. Partie I.*, Publications Mathematiques d'Orsay, 1984.

[19] Dwyer, J., Barnard, R. W., Cook, D., Corte, J., *Iteration of complex functions and Newton's method*, Austral. Sen. Math. J. **23** (2009), no. 1, 9–16

[20] Fatou, P., *Sur les équations fonctionelles*, Bull. Soc. Math. France **47** (1919), 161–271; **48** (1920), 33-94, 208-314

[21] Gilbert, W. J., *The complex dynamics of Newton's method for a double root*, Computers Math. Applic., **22** (1991) no. 10, 115–119, MR1136181 (92g:65056)

[22] Gilbert, W. J., *Newton's method for multiple roots*, Comput. & Graphics. **18** (1994) no. 2, 227–229

[23] Gilbert, W. J., *Generalizations of Newton's method*, Fractals, **9** (2001) no. 3, 251–262, MR1862380

[24] Haeseler, F. v., Peitgen, H.-O., *Newton's method and complex dynamical systems*, Acta Appl. Math. **13** (1988) no. 1-2, 3–58, MR0979816 (90a:58102)

[25] Julia, G., *Mémoire Sur l'itération des fonctions rationelles*, J. Math. Pures Appl. **8** (1918) no. 1, 47–245

[26] Keen, L., *Julia sets of rational maps*. (Complex dynamical systems, Cincinnati, OH, 1994), Proc. Sympos. Appl. Math., 49, Amer. Math. Soc., Providence, RI, 1994. 71–89, MR1315534

[27] Mandelbrot, B.B, *Fractals and chaos: the Mandelbrot set and beyond*, Springer-Verlag, New York, 2004

[28] Mañé, R., Sad, P., Sullivan, D., *On the dynamics of rational maps*, Ann. Sci. Éc. Norm. Supér. (4), **16** (1983) no. 2, 193–217

[29] McMullen, C., *Families of rational maps and iterative root-finding algorithms*, Ann. Math. (2), **125** (1987) no. 3, 467–493, MR0890160

[30] Milnor, J., *Dynamics in one complex variable*, Ann. of Math. Stud., vol 160, Princeton University Press, Princeton, N.J., 2006, MR2193309

[31] Myrberg, P. J., *Sur l'itération des polynomes r'eels quadratiques*, J. Math. Pures Appl. (9), **41** (1962) 339–351, MR0161968

[32] Ocken, S., *Convergence criteria for attracting cycles of Newton's method*, SIAM J. Appl. Math. **58** (1998), no. 1, 235–244, MR1610076 (99f:30011)

[33] Peitgen, H.-O., Richter, P. H., *The beauty of fractals*, Springer, Berlin, Heidelberg, New York and Tokyo, 1986

[34] Peitgen, H.O., Saupe, D., F.v.Haeseler, *Cayley's problem and Julia sets*, Math. Intelligencer **6** (1984), no. 2, 11–20, MR0738904 (85h:58101)

[35] Plaza, S., Romero, N., *Attracting cycles for the relaxed Newton's method*, J. Comput. Appl. Math., **235** (2011), no. 10, 3238–3244, MR2773308 (2012b:65064)

[36] Ritt, J., *On the iteration of rational functions*, Trans. Amer. Math. Soc. **21** (1920), 348–356

[37] Schröder, E., *Über unendlich viele algorithme zur Auflösung der Gleichungen*, Math. Ann., **2** (1870), no. 2. 317–365, MR1509664

[38] Schröder, E., *Über iterierte Funktionen*, Math. Ann., **3** (1871), no. 2, 296–322, MR1509704

[39] Shishikura, M., *The connectivity of the Julia set and fixed points*. (Complex dynamics, Wellesley, MA, 2009), 257–276, MR2508260 (2010d:37093)

[40] Stanewitz, R. L., Rolf, J. S., *Complex Dynamics: Chaos, Fractals, the Mandelbrot Set, and More*. (Explorations in Complex Analysis, Classr. Res. Mater. Ser., Math. Assoc. America, Washington, DC, 2012), 1-83, MR2963950

[41] Walsh, J. A., *The dynamics of Newton's method for cubic polynomials*, College Math. J., **26** (1995), no. 1, 22–28

[42] Xingyuan, W., Wei, L., *The Julia set of Newton's method for multiple root*, Appl. Math. Comput., **172** (2006), no. 1, 101-110, MR2197891

[43] Yang, W., *Symmetries of the Julia sets of Newton's method for multiple root*, Appl. Math. Comput. **217** (2010), no. 6, 2490-2494, MR2733691