Using stochastic differential equations to describe the transport of Cosmic Rays in space

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The origin of energetic particles in space remains a fundamental and outstanding problem in Astrophysics. Solutions to the Parker Transport equation descibe how such particles traverse the turbulent interplanetary magnetic field before arriving at Earth. This paper reviews a technique that numerically solves the Parker equation by exploiting its relationship with the theory of stochastic processes. Simulated proton energy spectra are shown to be in good agreement with energy spectra obtained via previous finite differencing methods. In addition, the stochastic integration technique reveals new information about the beahvior of individual particles as they traverse the heliosphere. Because of its generality, this method will allow one to tackle more complicated Cosmic Ray transport scenarios, and can be extended to describe other physical systems that are diffusive in nature.

Introduction

Cosmic rays are energetic particles that impinge upon the Earth's atmosphere. Discovered in 1912 by Austrian scientist Victor Hess (who received the 1936 Nobel Prize in Physics for this discovery), it was determined that they were of an extraterrestrial origin. Cosmic rays have been observed at a wide variety of energies up to 10^{21} eV. The observed particle flux (number of particles per unit time per unit area) at Earth roughly follows a power law, where $flux \approx E^{-3}$ (see Figure 1). This large variation in observed particle energies imply that they originate from a variety of sources. For example, cosmic rays with energies up to $\approx 10^{15}$ eV (corresponding to the "knee" in Figure 1) are believed to originate from supernova explosions across the Galaxy. Additionally, the steep "dip" found at energies less than $\approx 10^{9}$ eV is believed to arise because of a "sweeping out" effect the solar wind has on these lower-energy particles, thus making it difficult for them to penetrate into the inner solar system [1].

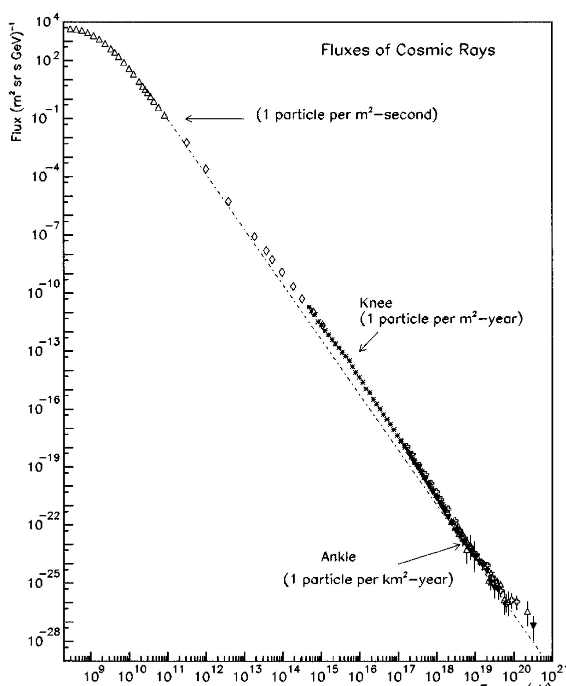

FIG. 1: The cosmic-ray energy spectrum as observed at Earth.

Discovering the origin of all possible sources of these cosmic rays is still a widely unanswered question in Astrophysics. While one can speculate on physical processes that are capable of producing such energetic particles, it is neccesary to understand how these particles gain (or lose) the appropriate amount of energy and momentum as they traverse the solar system to be observed on Earth. The equation that describes such a process is called the Parker Transport Equation, and it is the (numerical) method of solution that I consider in this paper.

This paper is outlined as follows. It begins with describing some background information regarding the discovery and nature of the turbulent interplanetary magnetic field. This is followed by a description/derivation of Parker's Transport Equation, as he first presented it in 1965. Next is discussion of a unique method of solution (stochastic integration) to this equation that was first presented by Jokippi in 1975. This technique was then formalized by Zhang in 1999, and a review of this paper is made. Appendices containing more detailed physics and mathematical derivations are attached and cited throughout when relevant.

Irregularities in the Interplanetary Magnetic Field

The Interplanetary Magnetic Field is the name for the magnetic field(s) permeating space within the Solar System. The Sun is responsible for the creation of this field. In the crudest sense, the Sun can be thought of as a large rotating ball of ionized gas. It is well known from basic electricity and magnetism that a rotating sphere of charged particles will create a magnetic field in the shape of a dipole [2].

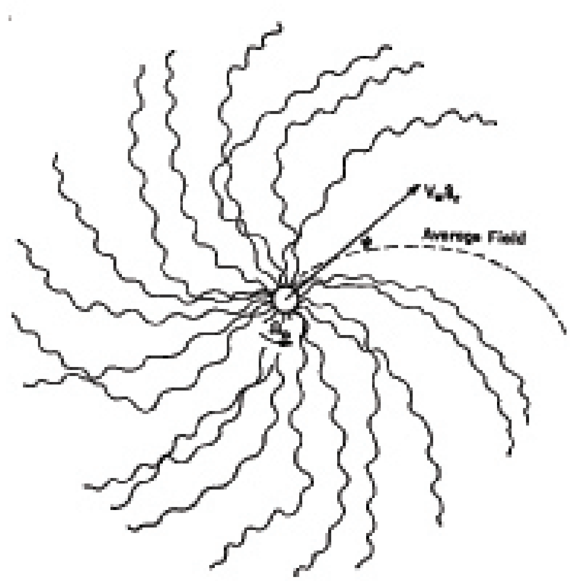

FIG. 2: A schematic view of the lines of force of the interplanetary magnetic field projected onto the solar equitorial plane. Image courtesy of[13] .

In 1951, observations of the motions of comet tails in the solar system suggested that large amounts of gas were streaming radially outward from the sun in all directions [3], with velocities around 500 - 1500 km/sec. These streams of gas were later named the solar wind. In 1958, E. N. Parker considered the effect these gas streams had on the magnetic field produced by the sun [4], by computing a steady-state magnetic field originating from a spherically symmetric gas outflow from a rotating star . He concluded that the ionized streams of gas "draw out" the lines of force associated with the solar magnetic field, in such a way so that the interplanetary magnetic field takes on a spiral shape. (See Figure 2). This spiral is often called the Parker spiral, or an Archemedian spiral in the literature. Furthermore, he predicted that because of certain dynamic instabilities [6] [7], the lines of force of the magnetic field would become "disorganized" at distances greater than 1 A.U.

In November of 1963, NASA launched the Explorer XVIII satellite, whose mission was to "study charged particles and magnetic fields in cislunar space" [8]. From these observations, Ness. et. al. [5] were able to verify that the shape of the interplanetary magnetic field was indeed a spiral. They also found that these magnetic field lines of force weren't smooth, but rather had small "irregularities" in them. These irregularities appear with a scale size around $10^5 - 10^7$ km., which is considered small compared to the overall dimensions of interplanetary space.

In 1964, E.N. Parker considered the effect of an magnetic irregularity on the motion of a charged particle [10]. By integrating the equations of motion of a charged particle moving in a uniform magnetic field with a small perturbation perpendicular to the uniform component, he showed that the particle will scatter (i.e., the pitch angle of the particle reverses rapidly and its phase angle changes randomly); furthermore, he found that the amount of scattering depended on the relative size of its gryoradius, r_G (see Appendix A), to the characteristic lengthscale of the irregularity/scatterer, L. Maximum scattering occurs for values of $\frac{r_G}{L} \approx 1$, and diminishes for ratios that deviate from 1. Since the gyroradious of the particle is related to its energy (or rigidity) (by definition, $r_G = \frac{mv_\perp}{qB}$), then this implies that particles at lower energies can more easily penetrate into the solar system – their motion can be treated by the guiding center approximation (see Appendix A), and they will either be reflected or transmitted without any significant change in pitch angle. Particles with the appropriate energies will be deflected as they traverse the interplanetary magnetic field.

1. THE PARKER TRANSPORT EQUATION

The Parker Transport equation is an equation that describes the time-evolution of a distribution of charged particles moving through a medium of electromagnetic fields that permeate space. It was initially written by E.N. Parker in 1965 [9], and has been widely used to model the propagation of cosmic rays through the solar system.

In 1964, Parker showed [10] that the presence of magnetic irregularities in the turbulent Interplanetary Magnetic Field can cause cosmic ray particles to scatter back and forth across the lines of force of the larger-scale field. When viewed from a large-scale perspective, he hypothesized that a group of scattering particles is equivalent to them undergoing a random-walk along, and across, the lines of force on which they are guided. Therefore, particle scatterings can be thought of as a diffusion process (see Appendix C).

Previous work by Chandrasekhar [11] showed that the random-walk of particles can be described by Markov stochastic process. [1] He showed that the equations of motion of such particles is equivalent to solving the Fokker-Planck equation. Parker took this reasoning and predicted that the random-walk of particles scattering along magnetic field irregularities should also be described by the Fokker-Planck equation. In this case, it's written as

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_i}(v_i W) - \frac{\partial}{\partial x_i}(\kappa_{ij}\frac{\partial W}{\partial x_j}) = 0 \tag{1}$$

Here, $W(x_i, t)$ represents the probability distribution of a charged particle as a function of time and its spatial position x_i; κ_{ij} is a tensor containing the diffusion coefficents for particles scattering in a given direction; and v_i represents the speed at which the magnetic irregularities travel (in this case, this is the solar wind speed, combined with any particle drift terms deemed neccesary) (See Appendix B for details on particle drifts).

Additionally, while the particle is riding along with the fields in the solar wind, the magnetic fields themselves are expanding because of the radial divergence of the wind. This causes the particle to be adiabatically cooled, so that it's kinetic energy, T, declines as

$$\frac{1}{T}\frac{dT}{dt} = -\frac{n(T)}{3}\frac{\partial v_i}{\partial x_i} \tag{2}$$

Here, $n(T)$ is a scale factor that depends on the average energy of the particle (e.g., $n(T) = 1$ for extreme relativistic particles, and $n(T) = 2$ for most non-relativisitic particles). [9]

[1] In this sense, a Markov process is a stocahstic process where the probability distribution of future states of a particle depend only on the the probability distribution of the previous state immediately before; in other words, it describes stochastic processes that have a limited memory.

To account for this energy decline, it's easier to consider the probability distribution over the kinetic energy of the particle. Let $U(x_i, T, t)$ represent that distribution, so that

$$W(x_i, t) = \int_0^\infty U(x_i, T, t) dT \tag{3}$$

The Fokker-Planck equation [1] can then be re-cast into the Parker Equation:

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x_i}(U v_i) + \frac{\partial}{\partial T}(U \frac{dT}{dt}) - \frac{\partial}{\partial x_i}(\kappa_{ij} \frac{\partial U}{\partial x_j}) = 0 \tag{4}$$

where the relationship for $\frac{dT}{dt}$ is given in Equation (2).

The Parker equation is commonly written in vector-notation (instead of index notation), and the probability distribution function, $f(\mathbf{x}, \mathbf{p}, t)$, is integrated over the particle momentum:

$$\frac{\partial f}{\partial t} = \boldsymbol{\nabla} \bullet ((\boldsymbol{\kappa} \bullet \boldsymbol{\nabla}) f - \mathbf{V} f) + \frac{1}{3}(\boldsymbol{\nabla} \bullet \mathbf{V}) \frac{1}{p^2} \frac{\partial}{\partial p}(p^3 f) \tag{5}$$

where $\mathbf{V}$ represents the speed at which the magnetic irregularities are traveling (often a combination of the local solar wind velocity and appropriate drift velocities); we've also used the fact that the adiabatic cooling rate, Equation (2), can be cast as a momentum decline, so that $\frac{1}{p}\frac{dp}{dt} = -\frac{1}{3}\frac{\partial v_i}{\partial x_i}$. [9]

For a diffusive process, there is no well-known prescription for determining the diffusion coefficent(s) other than from its definition [18]. A diffusion coefficent, D, is related to the average particle stepsize in a given direction (Δx) in a given time interval (Δt) by $D = \frac{(\Delta x)^2}{\Delta t}$ [11, 18] (Also, see Appendix C). Using this definition, let L denote the average stepsize the particle makes **along** the magnetic field B_i, and ν represent the number of steps taken per unit time. Then the diffusion coefficent along the field is given by $\kappa_\parallel \approx \nu L^2$. Similarly, the diffusion coefficent perpendicular to the field can be expressed as $\kappa_\perp \approx \nu r_G^2$, where r_G is the radius of gyration of the particle (see Appendix A). From this, Parker [9] deduced a general expression for the diffusion tensor κ_{ij}

$$\kappa_{ij} = \frac{\nu L^2}{\nu^2 + \Omega^2}[\nu^2 \delta_{ij} - \Omega i \Omega_j + \nu \epsilon_{ijk} \Omega_k] \tag{6}$$

where $\Omega_i = \frac{qB_i}{mc}$ is the cyclotron frequency around the field component B_i, and can be written in terms of the gyroradius and perpendicular velocity of the particle, $v_\perp = r_G \Omega_i$.

2. EARLY SOLUTIONS TO PARKER'S EQUATION USING STOCHASTIC INTEGRATION

The Parker Equation

$$\frac{\partial f}{\partial t} = \boldsymbol{\nabla} \bullet ((\boldsymbol{\kappa} \bullet \boldsymbol{\nabla}) f - \mathbf{V} f) + \frac{1}{3}(\boldsymbol{\nabla} \bullet \mathbf{V}) \frac{1}{p^2} \frac{\partial}{\partial p}(p^3 f) \tag{7}$$

is a partial differential equation for the (isotropic) distribution function, $f(\mathbf{x}, p, t)$, which is dependent on the spatial coordinate(s) and momentum. In 3 spatial dimensions, the distribution function is a function of 4 + 1 variables (3 spatial, 1 momentum, and time). There exist numerous numerical techniques to solve partial differential equations (e.g., finite-differencing or finite-element schemes), for which many have been used in the literature. Depending on the geometry involved and the expressions for the parameters in the equation, solutions can be difficult to obtain without making many simplifying assumptions.

An alternative method of solving this equation can be used by exploiting the random-walk behavior of each particle. This method is sometimes called "individual particle transport" , "Monte-Carlo simulation", or "stochastic integration." [12, 14, 15]

The first attempt at using the stochastic integration technique to solve Parker's Equation was made by J.R. Jokipii and A.J. Owens in 1975 [14]. They considered a simplified version of the Parker equation, by considering spherically symmetric solutions with zero particle drift, constant solar wind drift, and a single non-zero and constant diffusion coefficent, κ_{rr}. The version of the Parker equation they solved was given by

$$\frac{\partial f}{\partial t} = \kappa_{rr}\frac{\partial^2 f}{\partial r^2} - \frac{\partial}{\partial r}[(V + \frac{2\kappa_{rr}}{r})f] + \frac{\partial}{\partial T}[(\frac{4VT}{3r})f] \tag{8}$$

where $V \approx 350\frac{km}{s}$, T is the kinetic energy of the particle, and κ_{rr} the radial diffusion coefficent. They claimed that this equation is a form of a "simple diffusion equation", identical to the equation obtained from a 1 dimensional random walk (see Appendix C). By casting it into a more general form,

$$\frac{\partial f}{\partial t} = \frac{\partial^2}{\partial r^2}[\frac{(\Delta r_{diffusion})^2}{2\Delta t}f] - \frac{\partial}{\partial r}[(\frac{\Delta r_{convection}}{2\Delta t})f] + \frac{\partial}{\partial T}[(\frac{\Delta T}{2\Delta t})f] \tag{9}$$

they wrote down particle evolution equations

$$r_{n+1} = r_n \pm \sqrt{2\kappa_{rr}\Delta t} + [V + \frac{2\kappa_{rr}}{r}]\Delta t \tag{10}$$

$$T_{n+1} = T_n - (\frac{4VT}{3r})\Delta t \tag{11}$$

By creating a large number of particles at the origin $r = 0$ with some initial kinetic energy T_0 at time $t = 0$, they evolved the particles according to the random-walk equations above for a set timestep Δt. The distribution function, $f(r, T, t)$ was obtained by creating a three-dimensional histrogram of the corresponding values of radius, energy, and time.

This approach, while correct, has limited applications [12]. Depending on the form of the diffusion coefficent, it may not be possible to express the Parker Equation as a simple diffusion equation (i.e., cross-derivative terms may appear for an anisotropic diffusion coefficent). In this case, it will be neccesary to find a more general formulation for writing down the particle evolution equations for **any** form of the Parker Equation. Fortunately, it is possible to to do this, if one considers the connection between the Fokker-Planck equation and Ito stochastic differential equations (see Appendix D for a derivation of this equivalence).

3. A FORMALIZED APPROACH TO STOCHASTIC INTEGRATION

Because of the equivalence between the Fokker-Planck Equation and a corresponding Ito stochastic differential equation (SDE) (see Appendix D), one can reduce the problem of solving a multi-dimensional Fokker-Planck PDE to integrating a set of first order ordinary differential equations with a random component (an SDE). This is the approach that Zhang takes in his 1999 paper [12].

Zhang [12] noticed that the Parker Equation (7) can be transformed into a Fokker-Planck equation of the form

$$\frac{\partial P(t, \mathbf{q})}{\partial t} = \frac{1}{2}\Sigma_{i,j}\frac{\partial^2}{\partial q_i q_j}[G_{ij}P(t, \mathbf{q})] - \Sigma_i\frac{\partial}{\partial q_i}[K_i P(t, \mathbf{q})] \tag{12}$$

(where $G_{ij} = \Sigma_k B_{ik}B_{kj}$) by setting the 4-by-4 tensor, $\mathbf{G}$ to be

$$\begin{bmatrix} 2\boldsymbol{\kappa} & \mathbf{0} \\ 0 & 0 \end{bmatrix}$$

and the 4 dimensional vector, $\mathbf{K}_i$ to be

$$\begin{bmatrix} \mathbf{V} + \mathbf{V}_d + (\nabla \bullet \boldsymbol{\kappa}) \\ -\frac{1}{3}(\boldsymbol{\nabla} \bullet \mathbf{V})p \end{bmatrix}$$

With these transformations in-hand, Zhang wrote down the corresponding SDEs as

$$d\mathbf{X} = (\boldsymbol{\nabla} \bullet \boldsymbol{\kappa} - \mathbf{V} - \mathbf{V}_d)dt + \Sigma_k \boldsymbol{\alpha}_k d\mathbf{W}_k(t) \quad (13)$$

$$dP = \frac{1}{3}P(\boldsymbol{\nabla} \bullet \mathbf{V})dt \quad (14)$$

where $\mathbf{X}$ represents a 3-dimensional vector containing the spatial coordinates (e.g., $[x(t), y(t), z(t)]$), and $\Sigma_k \alpha_{ik}\alpha_{kj} = 2\kappa_{ij}$.

To see how this stochastic integration technique compares to other numerical techniques, Zhang chose to integrate a form of the Parker Equation that was solved (via the Alternating Direction Method) by Jokipii and Kopriva in 1979 [16]. The assumptions made for this form of the Parker Equation are as follows: The diffusion tensor is diagonal so that κ_{ij} is given by

$$\begin{bmatrix} \kappa_{\parallel} & 0 & 0 \\ 0 & \kappa_{\perp} & 0 \\ 0 & 0 & \kappa_{\perp} \end{bmatrix}$$

This simplifies the form for the $\boldsymbol{\alpha}$ variables, so that $\boldsymbol{\alpha}_1 = [\sqrt{2}\kappa_{\parallel}, 0, 0]$, $\boldsymbol{\alpha}_2 = [0, \sqrt{2}\kappa_{\perp}, 0]$, and $\boldsymbol{\alpha}_3 = [0, 0, \sqrt{2}\kappa_{\perp}]$. The diffusion coefficents $\kappa_{\parallel}$ and $\kappa_{\perp}$ were approximated by $\kappa_{\parallel} = (\kappa_{\parallel})_0\beta(\frac{p}{1GeVc^{-1}})^{b_{\parallel}}(\frac{B_e}{B})^{a_{\parallel}}$ and $\kappa_{\perp} = (\kappa_{\perp})_0\beta(\frac{p}{1GeVc^{-1}})^{b_{\perp}}(\frac{B_e}{B})^{a_{\perp}}$ (where β represents the fraction of particle speed relative to the speed of light, B_e is the magnetic field strength at the Earth, and B is the magnetic field at the location of the particle). The solar wind speed has only a radial component and is taken to be a constant. The drift velocity, $\mathbf{V}_d$ includes both gradient and curvature drifts (see Appendix B), and are computed from a Parker spiral equation for the interplanetary magnetic field

$$\mathbf{B} = \frac{A}{r^2}(\mathbf{e}_r - [\frac{r\Omega_{\odot}\sin(\theta)}{V}]\mathbf{e}_{\theta})[1 - 2H(\theta - \frac{\pi}{2})] \quad (15)$$

where r and θ are the heliospheric radius and angle, $\Omega_{\odot}$is the angular speed of the Sun's rotation, H is the Heaviside step function, and A is a constant that changes depending on the polarity of the solar magnetic field (which is known to periodically change sign). The drift velocity, $\mathbf{V}_d$ can be computed (see Appendix B) from the interplanetary magnetic field, $\mathbf{B}$, given above. However, the expression one obtains for $\mathbf{V}_d$ is cumbersome and contains singularities at heliospheric angles of $\theta = \frac{\pi}{2}$, which are deemed unphysical. For this reason (and to save on computation time) an approximation to $\mathbf{V}_d$ was found by [17], and is given by

$$V_d = (0.457 - 0.412\frac{|d|}{R_g} + 0.0915\frac{|d|^2}{R_G^2})v \quad (16)$$

where d represents the distance the particle is from the plane of the guiding center (the "current sheet"), and R_G is the gyro-radius of the particle. A list of the numerical values for the parameters described above are given in Figure 3. Zhang performed two simulations, labeled as Cases 1 and 2. Case 1 used the exact same numerical values as [16], and provided a direct comparison to the results obtained by Jokipii and Kopriva [16]. Case 2 used different numerical values that gave a "more realistic condition" of the conditions in the heliosphere (e.g., the diffusion coefficents are no longer constant, but vary with the strength of the magnetic field lines).

The numerical solution to Equation (13) was performed using a Monte-Carlo simulation of the equations. 3000 test particles were created with initial and final values given by Figure 3, and the random Wiener process terms, ΔW_i were approximated as Gaussian distributions with a 1σ width equal to the time increment Δt. The value for Δt was chosen "to be small enough for the simulated particles to have at least several tens of steps."

VALUES OF PARAMETERS USED IN THE SIMULATION FOR THE MODULATION OF PROTONS

Parameter	Case 1	Case 2
$\kappa_\parallel$	$50 \times 10^{20} \beta p^{1/2}$ cm^2 s^{-1}	$100 \times 10^{20} \beta p (B_e/B)$ cm^2 s^{-1}
$\kappa_\perp$	$5 \times 10^{20} \beta p^{1/2}$ cm^2 s^{-1}	$5 \times 10^{20} \beta p (B_e/B)$ cm^2 s^{-1}
V	400 km s^{-1}	400 km s^{-1}
$B_e = \sqrt{2}A/(\mathrm{AU})^2$	5nT	5nT
$T_\odot = 2\pi/\Omega_\odot$	27.27 days	27.27 days
R_b (outer boundary).......	10 AU	75 AU
r_0 (inner boundary).......	0.01 AU	0.01 AU
$f_{ism}(p)$ at $r = R_0$	$\propto (m_0^2 c^2 + p^2)^{-1.8}/p$	$\propto (m_0^2 c^2 + p^2)^{-1.8}/p$

FIG. 3: Numerical values used in Zhang's computation of the SDE's given in equation (13). Case 1 parameters are chosen to be exactly the same as those used in [16], whereas the Case 2 parameters are chosen to describe a more realistic situation. Image taken from [12].

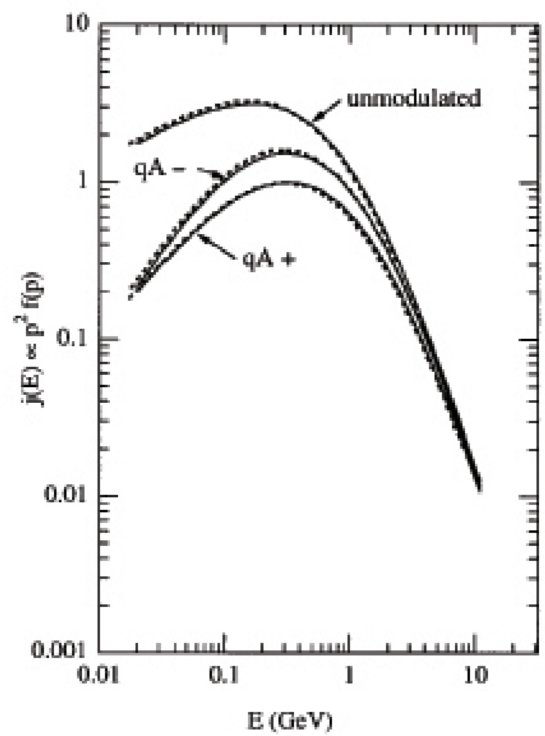

FIG. 4: Computed energy spectra at a distance of $r = 1$AU, for parameters given in Case 1 (see Figure 3). q represents the charge of the particle ($q = +1$ for protons simulated here), and A is a constant that describes the magnetic polarity of the solar magnetic field (known to flip its sign every 11 years). The unmodulated interstellar spectrum, $f_{ism}(p)$ is computed and displayed as a reference. The results obtained via stochastic integration in [12] are represented by solid lines, whereas the results obtained via the A.D.I. method in [16] are given by the dashed lines.

Figure 4 shows the results of the comparison with parameters given by Case 1 in Figure 3. Zhang computed the energy spectra for protons at a distance of 1 A.U. (the cosmic-ray distribution, $f(\mathbf{x}, p, t)$ is related to the energy spectrum, $j(\mathbf{x}, p, t)$, by $j(\mathbf{x}, p, t) = 4\pi p^2 f(\mathbf{x}, p, t)$), and compared it to that computed by Jokipii and Kopriva [16]. The results from each of these methods are in strong agreement, thereby giving confirmation that this method of stochastic integration does, in fact, produce reasonable results. In addition to computing energy spectra, information about individual particle trajectories were readily available, and are given in Figure 5.

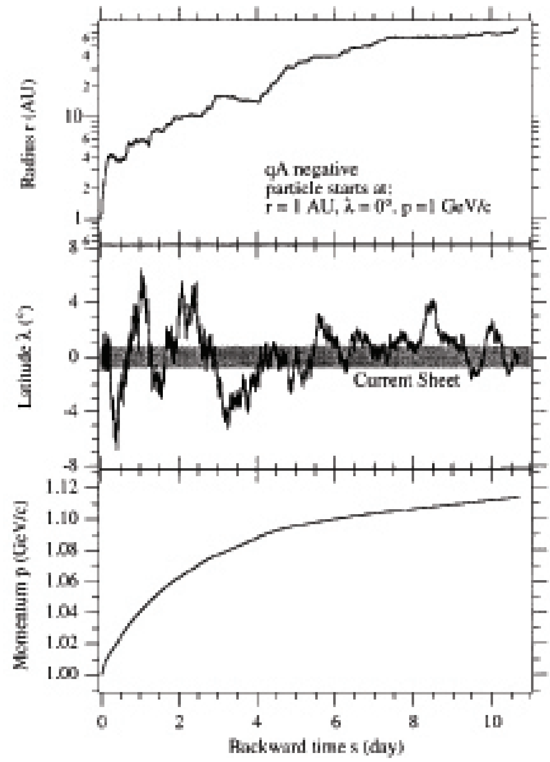

FIG. 5: A sample particle trajectory for a situation described by Case 2 in Figure 3. The particle started at a distance of $r = 1$ AU with momentum $p = 1$ GeV/c, and traversed the magnetic field until it reached a distance of $r = 75$AU. Above are plots of the particle's radial distance, latitude, and momentum as a function of time.

4. CONCLUSIONS

The study reviewed here presents an alternative method to numerically solving the Parker Transport Equation describing the propagation of cosmic-rays throughout the heliosphere. The main benefit of this stochastic integration method is that it allows one to takle very complicated transport scenarios that are difficult to solve otherwise using other standard numerical techniques, and, in certain applications, may speed up computation time considerably. The simulation can be done fully in 3 spatial dimensions and easily permits nearly any kind of geometric configuration for the solar wind plasma and heliospheric magnetic fields. Furthermore, stochastic integration provides information about the individual particle during their transit through the magnetic fields – particle trajectory, momentum loss or gains, and the transport time. None of these are readily available by other means of solving the Parker Equation.

Additionally, the stochastic integration technique described here can be extended to solve any equation that can be formulated with the Fokker-Planck equation. This is most useful, as the Fokker-Planck equation appears in many disciplines that encounter diffusive and advective processes.

5. APPENDIX A: THE GUIDING CENTER APPROXIMATION

Since cosmic-rays are charged particles that interact with the interplanetary magnetic field, it is useful to understand the basic physics of how charged particles move in such fields. One of the most fundamental postulates of physics states that electric and magnetic fields create forces on charged particles, so that their equations of motion can be determined from the Lorentz Force Law

$$m\frac{d\mathbf{v}}{dt} = \frac{q}{c}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{17}$$

Here, m, q, and $\mathbf{v}$ represent the particle's mass, charge, and velocity (a vector quantity), c is the speed of light (a constant), and $\mathbf{E}$ and $\mathbf{B}$ represent the electric and magnetic fields, respectively.

Let's first consider the equations of motion for a charged particle moving in a uniform magnetic field only. If we represent the particle's velocity by the vector $\mathbf{v} = v_x(t)\mathbf{e}_x + v_y(t)\mathbf{e}_y + v_z(t)\mathbf{e}_z$ and the magnetic field by $\mathbf{B} = B_0\mathbf{e}_z$ (where B_0 is a constant) then the equations of motion of the particle are just

$$\frac{dv_x}{dt} = -\Omega v_y \quad (18)$$
$$\frac{dv_y}{dt} = +\Omega v_x \quad (19)$$
$$\frac{dv_z}{dt} = 0 \quad (20)$$

where $\Omega = \frac{qB_0}{mc}$ and is named the cyclotron frequency. (It is also sometimes called the gyro-frequency, or the Larmor frequency). The solution to the above equations can be expressed as

$$v_x(t) = v\sin(\alpha)\cos(\Omega t - \phi) \quad (21)$$
$$v_y(t) = v\sin(\alpha)\sin(\Omega t - \phi) \quad (22)$$
$$v_z(t) = v\cos(\alpha) \quad (23)$$

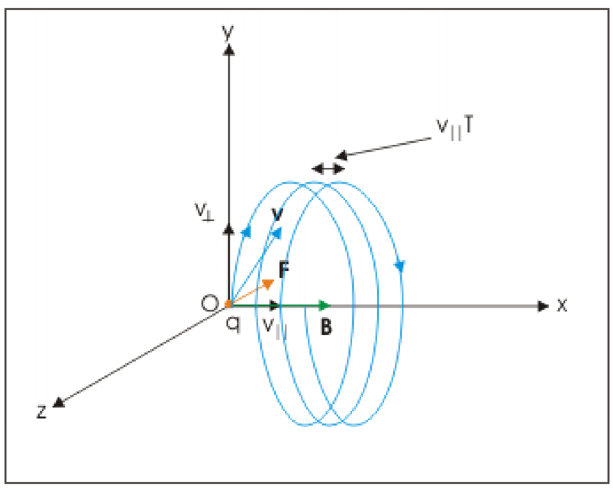

FIG. 6: A charged particle has a helical trajectory when placed in a uniform magnetic field. For a constant magnetic field pointing in a given direction, the particle will gyrate in circles on a plane perpendicular to the field line, while it's "guiding center" will move along the the direction of the field line. External forces will act on the particle so as to create a "drift velocity" term for the component parallel to the field line. (Image courtesy of http://cnx.org)

where v represents the magnitude of the particle's total velocity, α is called the pitch-angle, and ϕ is a phase angle. The pitch-angle is the angle of inclination of the particle's initial velocity from that of the magnetic field, which can be expressed as $\tan(\alpha) = \frac{v_\perp}{v_\parallel}$, where $v_\perp = \sqrt{v_x^2 + v_y^2}$ and $v_\parallel = v_z$. Thus, the particle's trajectory traces out a helical path – it moves in a circle at right angles to the field with a constant velocity parallel to the field (see Figure [6]). The radius of the circular motion is called the gyroradius (or Larmor radius), and is given by $r_G = \frac{v_\perp}{\Omega} = \frac{mv_\perp}{qB_0}$.

If we move to a frame of reference where $v_\parallel = 0$ (so, we move along with the speed of the guiding center in the direction of the magnetic field), we say that we are working in the Guiding Center System. The Guiding Center Approximation is when we decompose the particle motion into two components – the motion of the guiding center, and the gyromotion.

It is important to note that if one works in the Guiding Center reference frame, the circular motion of the charged particle will induce an electric current, and thus a magnetic moment (this is because of Faraday's Law). The magnitude of the magnetic moment is denoted by the symbol $\boldsymbol{\mu}$, and is defined to be

$$\boldsymbol{\mu} = \frac{q}{2}(\mathbf{r}_G \times \mathbf{v}_\perp) \tag{24}$$

which has magnitude $\mu = \frac{1}{2}\frac{q^2 r_G^2 B}{m} = \frac{1}{2}\frac{mv_\perp^2}{B} = \frac{W_\perp}{B}$, where $W_\perp$ represents the kinetic energy of the particle due to its perpendicular velocity.

6. APPENDIX B: PARTICLE DRIFTS

What is the trajectory of a charged particle under the influence of both an external force and a magnetic field? A charged particle will experience a velocity drift in the presence of an additional force, in addition to that elt from the magnetic field.

Consider a charged particle moving in a uniform magnetic field **and** a uniform electric field, so that $\mathbf{B} = B_0\mathbf{e}_z$ and $\mathbf{E} = E_0\mathbf{e}_x$ (B_0 and E_0 are constants). Inserting these into equation (17) gives the following equations of motion:

$$\frac{dv_x}{dt} = \Omega v_y + \frac{q}{mc}E_0 \tag{25}$$
$$\frac{dv_y}{dt} = -\Omega v_x \tag{26}$$
$$\frac{dv_z}{dt} = qE_0 \tag{27}$$

By making the substitution $v_y' \equiv v_y + \frac{E_0}{B_0}$, we get the same gyromotion trajectory as derived in Appendix A, only now the guiding center will drift away with speed $\frac{E_0}{B_0}$ in the y-direction, and the particle now accelerates along the direction of the magnetic field with magnitude qE_0.

This drift in the guiding center will hold true for **any** external force, as long as it has a component perpendicular to the magnetic field. Any component in the same direction as the magnetic field will accelerate the particle along the direction of the magnetic field. We can generalize the Lorentz Force law given in Equation [17], and consider the equations of motion perpendicular to the magnetic field:

$$m\frac{d\mathbf{v}_\perp}{dt} = \frac{q}{c}(\mathbf{v}_\perp \times \mathbf{B}) + \mathbf{F}_\perp \tag{28}$$

Since the external force will give rise to a drift velocity, we can transform coordinates, $\mathbf{v}_\perp = \mathbf{v}_{\perp'} + \mathbf{v}_D$ to find a general expression for the drift velocity, $\mathbf{v}_D$. Inserting this into Equation [28], and using the guiding center approximation that $m\frac{d\mathbf{v}_\perp}{dt} = \frac{q}{c}(\mathbf{v}_\perp \times \mathbf{B})$, we get the general expression for the drift velocity of the guiding center of a charged particle in an external force:

$$\mathbf{v}_D = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2} \tag{29}$$

6.1. Motion in an inhomogeneous magnetic field and the Gradient Drift

Consider now a particle in a slightly inhomogeneous magnetic field (so that the strength of the magnetic field is now a function of spatial coordinates, $B(x, y, z)$, or $B(\mathbf{r})$). If the variation of the magnetic field is smaller than the value of the average of the field itself, then the particle will experience an "external" force due to the gradients of the magnetic field itself. This external force gives rise to a velocity drift, which is often called the "gradient drift" in the literature.

We begin by expanding the magnetic field, $\mathbf{B}(\mathbf{r})$, into a Taylor series approximation and neglecting higher order terms (assuming that the magnetic field varies slowly):

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + (\mathbf{r} \bullet \nabla)\mathbf{B} + ... \tag{30}$$

Because we're expanding a vector quantity, $\mathbf{B}(\mathbf{r})$, that is a function of another vector, $\mathbf{r}$, the quantity that represents the gradient of $\mathbf{B}$ with respect to the position vector, $\mathbf{r}$, will be a dyad [2].

Inserting this into the Lorentz Force law, Equation [17], we get

$$m\frac{d\mathbf{v}}{dt} = \frac{q}{c}(\mathbf{v} \times \mathbf{B}_0) + \frac{q}{c}(\mathbf{v} \times (\mathbf{r} \bullet \nabla\mathbf{B})) \tag{31}$$

The last term on the right-hand side can be thought of as an "external" force:

$$\mathbf{F}_\perp = \frac{q}{c}(\mathbf{v}_\perp \times (\mathbf{r} \bullet \nabla)\mathbf{B}) \tag{32}$$

We can re-write this force term as

$$\mathbf{F}_\perp = \frac{q}{c}((\mathbf{v}_\perp \times \mathbf{r}) \bullet \nabla\mathbf{B}) \tag{33}$$

Noting that the $(\mathbf{v}_\perp \times \mathbf{r})$ term is just the definition of the magnetic moment, $\boldsymbol{\mu}$ (Appendix A), we can write the gradient force as

$$\mathbf{F}_\perp = -(\boldsymbol{\mu} \bullet \nabla)\mathbf{B} \tag{34}$$

Inserting this into Equation [29] gives us the drift velocity of the guiding center – the gradient drift:

$$\mathbf{v}_G = -\frac{\frac{1}{2}mv_\perp^2}{qB^3}(\nabla\mathbf{B}) \times \mathbf{B} \tag{35}$$

6.2. The Curvature Drift

In addition to non-zero spatial gradients, an inhomogeneous magnetic field may also have curvature associated with it ($\boldsymbol{\nabla} \times \mathbf{B} \neq 0$). The Guiding Center reference frame will then **not** be an intertial frame, since the curvature of the field lines will create a centrifugal force on the particle. The centrifugal force acting on the particle, $\mathbf{F}_c$, as seen from this non-inertial frame can be approximated (see [19]) by

$$\mathbf{F}_c = -mv_\parallel^2\frac{(\mathbf{B} \bullet \boldsymbol{\nabla}) \times \mathbf{B}_\perp}{B^2} \tag{36}$$

The drift velocity associated with this "external" force is found by inserting $\mathbf{F}_c$ into Equation (29).

$$\mathbf{v}_c = -\frac{mv_\parallel^2}{qB^4}[(\mathbf{B} \bullet \boldsymbol{\nabla})\mathbf{B}] \times \mathbf{B} \tag{37}$$

The direction of this drift velocity in the same direction as that of the gradient drift; they point perpendicular to the gyromotion plane of the magnetic field line.

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[2] A dyad is an operator represented by a pair of vectors which act on a vector. The dot product of a dyad with another vector returns a vector. For example, define the dyad $D = \mathbf{AB}$, and some vector of appropriate size $\mathbf{X}$. Then $D \bullet \mathbf{X} = \mathbf{A}(\mathbf{B} \bullet \mathbf{X}) = (\mathbf{B} \bullet \mathbf{X})\mathbf{A} = \alpha\mathbf{A}$, where α is a scalar quantity. The components of a dyad are often represented in a 3×3 matrix. D can be written as $A_iB_j\mathbf{e}_i\mathbf{e}_j$ (where $i, j = 1, 2, 3$). The scalar product with a vector $\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3$ is just $B_1X_1A_1\mathbf{e}_1 + B_2X_2A_2\mathbf{e}_2 + B_3X_3A_3\mathbf{e}_3$.

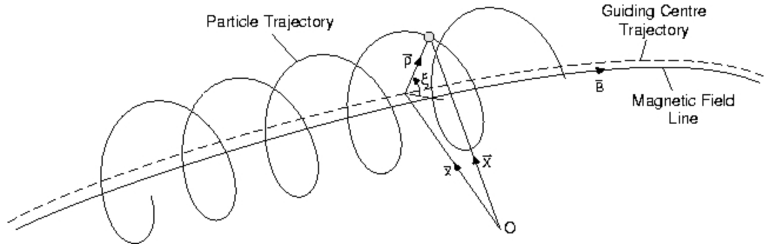

FIG. 7: A charged particle moving in a magnetic field will gyrate about an axis parallel to the magnetic field. External forces will cause the guiding center to undergo various drift velocities, in a direction perpendicular to the magnetic field lines. Such drift velocities are due to the inhomogeneities of the magnetic field (gradient drift), the curvature of the magnetic field lines (curvature drift), and the presence of an additional electric field ($\mathbf{E}\times\mathbf{B}$ drift).

7. APPENDIX C: A RANDOM-WALK DERIVATION OF THE DIFFUSION EQUATION

In this appendix, I show how one can relate the evolution of a random-walk of a particle in 1-dimension to the diffusion equation in 1D. While in no way rigorous, I think it's a nice way of illustrating the basic physics behind a random-walk, and showing where the diffusion coefficent originates from. This derivation was taken from [18].

Let $N(x,t)$ represent the number of particles at position x, and time t, and $N(x+\delta,t)$ represent the number of particles at position $x+\delta$ and time t. At time $t+\tau$, let's say that half of the particles at position x move to the right by an amount δ, the other half of the particles at x move to the left by δ, and that every particle at every gridspace along x follows this rule.

Now imagine a screen is placed in-between the positions x and $x+\delta$. The total number of particles that will cross to the right of the screen (defining right to be in a positive direction) is just $\frac{1}{2}N(x,t)-\frac{1}{2}N(x+\delta,t)$. Now, defining the particle flux through that screen to be J_x (and having units of particle number per time per area), and assigning the area of the screen to be A, we can write the net flux of particles crossing that screen to be:

$$J_x(x,t)=\frac{-[N(x+\delta,t)-N(x,t)]}{2A\tau} \tag{38}$$

We can rewrite this in terms of particle density ($n(x,t)\equiv\frac{N(x,t)}{A\delta}$) as

$$J_x(x,t)=-D\frac{[n(x+\delta,t)-n(x,t)]}{\delta} \tag{39}$$

where $D=\frac{\delta^2}{2\tau}$ is labeled the diffusion coefficent. In the limit as $\delta\to 0$, the particle flux is just

$$J_x(x,t)=-D\frac{\partial n}{\partial x} \tag{40}$$

Now consider a box (instead of a screen) placed on the grid, with boundaries at x and $x+\delta$. In a single timestep, τ, the number of articles that enter from the left (from the side of position x) is just $J_x(x,t)A\tau$, and the number of particles that leave the box from the right (from the side of position $x+\delta$) is $J_x(x+\delta,t)$. Therefore, the total increase in the number of particles in the box after a single timestep is

$$N(x,t+\tau)-N(x,t)=A\tau[J_x(x,t)-J_x(x+\delta,t)] \tag{41}$$

Rewriting this in terms of the particle density, $n(x,t)$ gives

$$\frac{n(x,t+\tau)-n(x,t)}{\tau}=-\frac{[J_x(x+\delta,t)-J_x(x,t)]}{\delta} \tag{42}$$

In the limit as $\tau\to 0$ and $\delta\to 0$, we have

$$\frac{\partial n}{\partial t} = -\frac{\partial J_x}{\partial x} \tag{43}$$

Substituting in with equation [40], gives us the 1 dimensional diffusion equation:

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2} \tag{44}$$

In equation [40], the diffusion coefficent is related to the grid sizes in time and space by:

$$D \equiv \frac{\delta^2}{2\tau} = \frac{(\Delta x)^2}{2\Delta t} \tag{45}$$

This can be re-written as

$$(\Delta x) = \pm\sqrt{2D\Delta t} \tag{46}$$

which can be placed into a discretized form as

$$x_{i+1} = x_i \pm \sqrt{2D\Delta t} \tag{47}$$

8. APPENDIX D: THE CONNECTION BETWEEN STOCHASTIC DIFFERENTIAL EQUATIONS AND THE FOKKER-PLANCK EQUATION [20]

A stochastic differential equation (SDE) is an ordinary differential equation that contains a term representing a fluctuating, random function of time. The simplest form of an SDE is given by the Langevin equation,

$$\frac{dx}{dt} = a(x,t) + b(x,t)\zeta(t) \tag{48}$$

where x is the variable of interest, $a(x,t)$ and $b(x,t)$ are certain known functions, and $\zeta(t)$ represents the randomly fluctuating component that depends only on time, t. In a mathematical sense, "randomly fluctuating", means that $\zeta(t)$ and $\zeta(t')$ are statistically independent (i.e., no correlation at different times). This implies that $<\zeta(t)>= 0$ and $<\zeta(t)\zeta(t')>= \delta(t-t')$.

Integrating directly for $x(t)$ would give

$$x(t) - x(0) = \int_0^t a[x(s),s]ds + \int_0^t b[x(s),s]dW(s) \tag{49}$$

where $dW(t) \equiv \zeta(t)dt$ is called the differential of the Weiner process, $W(t)$. [3] Any integral of the type $\int_{t0}^t G(t')dW(t')$ (where $G(t)$ is some arbitrary well-defined function) is called a stochastic Riemann-Stieltjes integral. In the same way that integrals from ordinary calculus are computed from sums of their Riemann summation counterparts, the Riemann-Stieltjes integrals can be computed. There are two different ways of computing such an integral, depending on how the time intervals are partitioned; these are called Ito and Stratonovich stochastic integrals. Mathematically, Stratonovich integrals are very difficult to compute in practice. For this reason, SDEs are often carried out using the Ito rules for integration.

[3] Physically, the Wiener process $W(t)$ can be thought of as the simplest 1D diffusion equation with zero drift and diffusion coefficent equal to $\frac{1}{2}$.

Therefore, a stochastic quantity $x(t)$ obeys an Ito SDE written as

$$dx(t) = a[x(t), t]dt + b[x(t), t]dW(t) \tag{50}$$

if for all t and t_0

$$x(t) = x(t_0) + \int_{t_0}^{t} a[x(t'), t']dt' + \int_{t_0}^{t} b[x(t'), t']dW(t') \tag{51}$$

This equation can be discretized according to

$$x_{i+1} = x_i + a(x_i, t_i)\Delta t_i + b(x_i, t_i)\Delta W_i \tag{52}$$

where

$$x_i = x(t_i) \tag{53}$$
$$\Delta t_i = t_{i+1} - t_i \tag{54}$$
$$\Delta W_i = W(t_{i+1}) - W(t_i) \tag{55}$$

Now, consider an arbitrary function of $x(t)$, $f[x(t)]$. What kind of SDE does it obey? This can be determined by expanding the function $f[x(t)]$ to second order, so that

$$df[x(t)] = f[x(t) + dx(t)] - f[x(t)] \tag{56}$$
$$= f'[x(t)]dx(t) + \frac{1}{2}f''[x(t)]dx(t)^2 \tag{57}$$
$$= f'[x(t)][a[x(t), t]dt + b[x(t), t]dW(t)] + \frac{1}{2}f''[x(t)]b[x(t), t]^2 dW(t)^2 \tag{58}$$

where the ' denotes the derivative with respect to the stochastic variable, x. Using the property that $dW(t)^2 = dt$, [20] we get

$$df[x(t)] = (a[x(t), t]f'[x(t)] + \frac{1}{2}b[x(t), t]^2 f''[x(t)])dt + b[x(t), t]f'[x(t), t]dW(t) \tag{59}$$

This formula is known as Ito's formula and it shows that changing stochastic variables isn't the same as the chain rule from ordinary calculus.

The connection to the Fokker-Planck equation arises when one considers the time development of the average of an arbitrary function that depends on a stochastic variable, $f(x(t))$. Using Ito's formula above (equation (59)), take the average of $f(x(t))$ and then the derivative $\frac{d}{dt}$, to get

$$\frac{d}{dt} < f[x(t)] > = \tag{60}$$
$$= < a[x(t), t]\partial_x f + \frac{1}{2}b[x(t), t]^2 \partial_x^2 f > \tag{61}$$

However, averages of stochastic variables and functions are computed with respect to a transition probability density, $p(x, t)$, so that $< f(x(t)) > \equiv \int f(x)p(x, t)dx$. Applying this to the above gives

$$\int f(x)\partial_t p(x, t)dx = \int [a[x(t), t]\partial_x f + \frac{1}{2}b[x(t), t]^2 \partial_x^2 f]p(x, t)dx \tag{62}$$

Integrating by parts and discarding surface terms for the right hand side gives

$$\int f(x)\partial_t p(x,t)dx = \int [-\partial_x(a[x(t),t]p(x,t)) + \frac{1}{2}\partial_x^2(b[x(t),t]^2 p(x,t))]f(x)dx \tag{63}$$

Since the above holds true for any arbitrary function, $f(x)$, then

$$\partial_t p(x,t) = -\partial_x[a(x,t)p(x,t)] + \frac{1}{2}\partial_x^2[b(x,t)^2 p(x,t)] \tag{64}$$

This is the Fokker-Planck equation, which has an equivalence to a diffusion process with diffusion coefficent $b(x,t)^2$ and drift coefficent $a(x,t)$. Any function that satisfies this partial differential equation will have a corresponding SDE given by equation (50), which can be discretized on a grid given by equations (52) and (53).

For the case with $a(x,t) = 0$ and $b(x,t) = D = constant$, the Fokker-Planck equation above reduces to the classical 1-D diffusion equation. The corresponding discretized SDE to this equation is given by $x_i = x_i + \sqrt{D}\Delta W_i$. It's interesting to note that this is the same equation and discretization as that derived in Appendix C (which was derived from a less rigorous, but more "physical" point of view). The advantage of using the Ito SDE formulation is that it allows one to write down the corresponding SDEs for more complicated drift and diffusion processes and in any number of dimensions. When the "physical" picture becomes too complicated to visualize, the mathematical perspective is a useful tool.

In general, SDEs that correspond to many variable systems can be defined for n variables by

$$d\mathbf{x}(t) = \mathbf{A}(\mathbf{x},t)dt + \mathbf{B}(\mathbf{x},t)d\mathbf{W}(t) \tag{65}$$

where $\mathbf{A}(\mathbf{x},t)$ is vector of length n, $\mathbf{B}(\mathbf{x},t)$ is an $(n \times n)$ matrix, and $d\mathbf{W}(t)$ is an n variable Weiner process. The corresponding Fokker-Planck equation is then

$$\partial_t p(\mathbf{x},t) = -\Sigma_i \partial_i[\mathbf{A}(\mathbf{x},t)p(\mathbf{x},t)] + \frac{1}{2}\Sigma_{i,j}\partial_i\partial_j[[\mathbf{B}(\mathbf{x},t)\mathbf{B}^T(\mathbf{x},t)]_{ij}p(\mathbf{x},t)] \tag{66}$$

where ∂_i and ∂_j denote differentiation with respect to the appropriate variable (e.g., $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, and $\partial_3 = \partial_z$ for a 3 dimensional system).

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