The Prime Number Theorem

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Introduction

The discipline which uses analysis to prove properties of natural numbers is called Analytic Number Theory. In this work I present one of the greatest achievements of this discipline; the proof of the Prime Number Theorem which in one of its many forms states that

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} = 1 \tag{1}$$

where $\pi(x)$ is the number of prime numbers less than or equal to $x \in \mathbb{R}$.

The various attempts to prove this theorem made Analytic Number Theory evolve into the powerful tool which it has come to be in the study of natural numbers, and in the meantime they unexpectedly motivated advancements in the field of Complex Analysis.

Euler was the first known mathematician to use the tools of analysis to prove theorems on number theory in his now famous proof of the divergence of the series $\sum_p 1/p$ made up of the reciprocals of prime numbers (from which the infinitude of the prime numbers follows). This proof used the divergence of the series $\sum_{n=1}^{\infty} 1/n$ and a remarkable relation which Euler himself had discovered between the prime numbers and the series $\sum_{n=1}^{\infty} 1/n^s$ where s is real and $s > 1$.

After subsequent generalizations by Euler and other mathematicians such as Dirichlet (from whom the received their name), a coherent and very interesting theory on Dirichlet series has been developed which will be presented in the first chapter of the present work. Proofs of the main theorems are given as well as various properties of these series which show the existing relations between them and the multiplicative structure of natural numbers, particularly with prime numbers. An amazing link between classical number theory and analysis, apparently distant disciplines in the mathematical world appears through these series.

At the end of the chapter some applications of this theory to classical number theory are developed. In particular,to the theory of convolutions of arithmetic functions. Results and definitions about convolutions are more transparent under this general context. For example, a clear motivation for the definition of the *Möbius* μ function will be given as well as a very short and simple proof of the *Möbius* inversion formula.

Chapters 2 and 3 are devoted mainly to the proof of the Prime Number Theorem. Chapter 2 deals with the Riemann ζ (zeta) function which is by far the most known Dirichlet series and of fundamental importance for the proof. Proofs of the main properties of this function needed for the proof of the Prime Number Theorem will be given. The Prime Number Theorem is proved in chapter 3.

The strategy of the proof of the Prime Number Theorem present in this work is mainly due to Riemann who in his revolutionary and only paper on number theory showed a possible line of attack to study $\pi(x)$ using the theory of Complex Analysis which he had been developing. In this article, Riemann introduces the ζ function and states his famous hypothesis, still without proof. The specific form of the proof is product of later simplifications and alternatives which have been found to elude the difficulties which Riemann encountered and is in essence the one exposed in [Jam03].

The Prime Number Theorem, which basically states the existence of some sort of regularity in the growth $\pi(x)$ function was stated for the first time (independently) by Legendre and Gauss in forms different from the one presented in (1). It finally proved (independently) by Hadamard and de la Valle Poussin in 1896, nearly forty years after Riemann's article had been published. Until 1949 all known proofs of the Prime Number Theorem used complex analysis and followed Riemann's ideas. In this year Selberg and Erdös found proofs which did not use complex analysis although they still used analysis. These proofs are nowadays called elementary. Proofs which use complex analysis are however, more transparent.

All the theorems present in this work can be found in the bibliography but the organization of the themes and theorems is my own. Some minor modifications and simplifications were made in various proofs and great stress was made to motivate the ideas behind some arguments. Unlike texts like [Apo76] by Apostol, this work is devoted completely to the proof on the Prime Number Theory. In some cases, more general results than needed are presented and proved when they result enlightening. In particular, a considerable amount of theory on Dirichlet series, ignored on some proofs of the prime number theorem where only the Riemann ζ function is studied, is presented to stress the existence of the relationship between these series and prime numbers.

Chapter 1

Dirichlet Series

This chapter will introduce Dirichlet series, very important series in analytic number theory. They are a generalization of the Riemann Zeta function.

1.0.1 Definition. Let $s \in \mathbb{C}$. A series of the form

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} \tag{1.1}$$

where $f : \mathbb{N} \to \mathbb{C}$ (different from zero for some n) is called a *Dirichlet series.*

The shorter notation $\sum f(n)/n^s$ for (1.1) will be used where there is no risk of confusion.

It is worthwhile to remind that for $s \in \mathbb{C}$ and $n \in \mathbb{N}$, n^s is defined by $n^s = e^{s \log n}$ where $\log n$ is the real and natural logarithm of n. This function ($s \to n^s \in \mathbb{C}$) is analytic in the whole complex plane and has derivative $n^s \log n$. The notation $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$ will be used systematically throughout this work to denote an arbitrary complex number. It is to be understood that under this convention when writing s_0 it will follow that $s_0 = \sigma_0 + it_0$ where $\sigma_0, t_0 \in \mathbb{R}$. Note also that $|n^s| = n^\sigma$ for any $s \in \mathbb{C}$.

The most famous Dirichlet series is $\sum 1/n^s$, known as the ζ function and defined for s where the series converges. It was studied for the first time by Euler (who discovered its relation with prime numbers) as a function of real variable and named ζ by Riemann, the first one to study it as a function of a complex variable. Dirichlet series where studied systematically by Dirichlet who generalized the results discovered by Euler on the ζ function to prove the famous theorem on number theory which now bears his name.

1.1 Analytic Properties

1.1.1 Lemma. *If there exist $s_0 \in \mathbb{C}$ and $M \in \mathbb{R}$ such that*

$$\left| \sum_{n \le x} \frac{f(n)}{n^{s_0}} \right| \le M \qquad \forall x \ge 1$$

then, for all $s \in \mathbb{C}$ with $\sigma > \sigma_0$ and for $a, b \in \mathbb{N}$ with $0 < a < b$ one has the following inequality

$$\left| \sum_{a < n \le b} \frac{f(n)}{n^s} \right| \le \frac{2M}{a^{\sigma - \sigma_0}} \left(1 + \frac{|s - s_0|}{\sigma - \sigma_0} \right).$$

Proof. Using Abel's identity B.1 (in the appendix) B with $a(n) = f(n)/n^{s_0}$ to the function $g(x) = 1/x^{s-s_0}$ (Note: $A(x) = \sum_{n \le x} f(n)/n^{s_0}$), one obtains

$$\sum_{a < n \le b} \frac{f(n)}{n^s} = A(b)b^{s_0 - s} - A(a)a^{s_0 - s} - (s_0 - s) \int_a^b A(t)t^{s_0 - s - 1} dt.$$

Therefore,

$$
\begin{aligned}
\left|\sum_{a<n\le b}\frac{f(n)}{n^s}\right| &\le Mb^{\sigma_0-\sigma}+Ma^{\sigma_0-\sigma}+\left|\int_a^b (s_0-s)A(t)t^{s_0-s-1}dt\right|\\
&\le 2Ma^{\sigma_0-\sigma}+\int_a^b |s_0-s|\,|A(t)|\,t^{\sigma_0-\sigma-1}dt\\
&\le 2Ma^{\sigma_0-\sigma}+|s_0-s|\,M\int_a^b t^{\sigma_0-\sigma-1}dt\\
&\le 2Ma^{\sigma_0-\sigma}+|s_0-s|\,M\int_a^\infty t^{\sigma_0-\sigma-1}dt\\
&= 2Ma^{\sigma_0-\sigma}+\frac{M\,|s_0-s|\,(-a^{\sigma_0-\sigma})}{(\sigma_0-\sigma)}\\
&= 2Ma^{\sigma_0-\sigma}\left(1+\frac{|s-s_0|}{2(\sigma-\sigma_0)}\right)\\
&\le 2Ma^{\sigma_0-\sigma}\left(1+\frac{|s-s_0|}{\sigma-\sigma_0}\right).
\end{aligned}
$$

■

1.1.2 Theorem. *For every Dirichlet series $\sum f(n)/n^s$ the exist $\sigma_c,\sigma_a\in\mathbb{R}$ with $-\infty\le\sigma_c\le\sigma_a\le\infty$ such that:*

1. *$\sum f(n)/n^s$ converges if $\sigma>\sigma_c$ and diverges if $\sigma<\sigma_c$.*
2. *$\sum f(n)/n^s$ converges absolutely if $\sigma>\sigma_a$ and does not if $\sigma<\sigma_a$.*
3. *$\sigma_a-\sigma_c\le 1$.*

Proof. If $\sum f(n)/n^s$ diverges on the whole plane take $\sigma_c=\sigma_a=\infty$ and if it converges on the whole plane take $\sigma_c=\sigma_a=-\infty$ (the reason for being able to take $\sigma_a=-\infty$ will be clear by what follows). Suppose none of the previous cases takes place and take s_0 where $\sum f(n)/n^{s_0}$ converges and $s\in\mathbb{C}$ with $\sigma>\sigma_0$. By lema 1.1.1 it follows that for $m_1,m_2\in\mathbb{N}$ with $0<m_1<m_2$,

$$
\left|\sum_{n=1}^{m_2}\frac{f(n)}{n^s}-\sum_{n=1}^{m_1}\frac{f(n)}{n^s}\right|=\left|\sum_{m_1<n\le m_2}\frac{f(n)}{n^s}\right|\le\frac{K}{m_1^{\sigma-\sigma_0}} \tag{1.2}
$$

where K is independent of m_1 and m_2. Therefore, given that $\lim_{m_1\to\infty}1/{m_1}^{\sigma-\sigma_0}=0$, the series of partial sums is a Cauchy series and so $\sum f(n)/n^s$ converges. It is the clear that the sought for σ_c is precisely

$$
\sigma_c=\inf\left\{\sigma\in\mathbb{R}\,\middle|\,\sum f(n)/n^\sigma \textit{ converges}\right\}.
$$

(Note that the set is bounded since $\sum f(n)/n^s$ does not converge on the whole of $\mathbb{C}$.)

Take now $s\in\mathbb{C}$ with $\sigma>\sigma_0+1$. Since $\sum f(n)/n^{s_0}$ converges, $\lim_{n\to\infty}f(n)/n^{s_0}=0$ and so there exists $M\in\mathbb{R}$ such that $|f(n)/n^{s_0}|\le M$ for all n. This implies that $|f(n)|\le Mn^{\sigma_0}$, and that

$$
\left|\frac{f(n)}{n^s}\right|=\frac{|f(n)|}{n^\sigma}\le\frac{Mn^{\sigma_0}}{n^\sigma}=\frac{M}{n^{\sigma-\sigma_0}}\qquad\text{with }\sigma-\sigma_0>1.
$$

It follows that $\sum f(n)/n^s$ converges absolutely since $\sum M/n^{\sigma-\sigma_0}$ converges by B.4 given that $\sigma-\sigma_0>1$.

Likewise, one can see that if $\sum f(n)/n^{s_0}$ converges absolutely, so does $\sum f(n)/n^s$ with $\sigma>\sigma_0$, given that

$$
\left|\frac{f(n)}{n^s}\right|=\frac{|f(n)|}{n^\sigma}\le\frac{|f(n)|}{n^{\sigma_0}}=\left|\frac{f(n)}{n^{s_0}}\right|.
$$

Therefore, one can take

$$
\sigma_a=\inf\left\{\sigma\in\mathbb{R}\,\middle|\,\sum f(n)/n^\sigma\text{ converges absolutely }\right\}
$$

and (2) follows.

(3) follows from the fact that convergence at s_0 implies absolute convergence at s for all $s \in \mathbb{C}$ with $\sigma > \sigma_0 + 1$. ■

The previous theorem shows that for every Dirichlet series which converges at some $s \in \mathbb{C}$ there exists a half plane $Re(s) > \sigma_c$ at which the series converges. In this half plane of convergence the series represents a function of the complex variable s

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \qquad Re(s) = \sigma > \sigma_c. \tag{1.3}$$

The next theorem shows that convergence in actually uniform in every compact subset of the half plane of convergence. From this fact it will follow that $F(s)$ is analytic.

1.1.3 Theorem. *The convergence of $\sum f(n)/n^s$ is uniform in every compact subset of the half plane of convergence $\sigma > \sigma_c$.*

Proof. Let $D = [a, b] \times [c, d]$ be with $a > \sigma_c$. Let $s \in D$ and $\sigma_0 \in \mathbb{R}$ with $\sigma_c < \sigma_0 < a$. By the lema 1.1.1 it follows that

$$\left| \sum_{m_1 < n \leq m_2} f(n)/n^s \right| \leq 2Mm_1^{\sigma_0 - \sigma} \left(1 + \frac{|s - \sigma_0|}{\sigma - \sigma_0}\right).$$

Since $m_1^{\sigma_0 - \sigma} \leq m_1^{\sigma_0 - a}$ and there exists a constant $A \in \mathbb{R}$ with $|s - \sigma_0| < A$ for all $s \in D$, one obtains

$$\left| \sum_{m_1 < n \leq m_2} f(n)/n^s \right| \leq 2Mm_1^{\sigma_0 - a} \left(1 + \frac{A}{a - \sigma_0}\right) = Bm_1^{\sigma_0 - a}$$

where B is independent of s. Given that $\lim_{m_1 \to \infty} m_1^{\sigma_0 - a} = 0$, the series of partial sums is *uniformly* Cauchy and so the series converges uniformly. ■

1.1.4 Corollary. *The function represented by the Dirichlet series (1.3) is an analytic function of the complex variable s in its half plane of convergence and*

$$F'(s) = -\sum_{n=1}^{\infty} \frac{f(n) \log n}{n^s} \tag{1.4}$$

where $\log n$ *is the real (and natural) logarithm of* n.

Moreover, the expression as Dirichlet series of $F'(s)$ (1.4) has the same half plane of absolute convergence as $F(s)$.

Proof. Given that $f(n)/n^s$ is an entire function (analytic in the whole complex plane), and the Dirichlet series is locally uniformly convergent in its half plane of convergence, $F(s)$ is analytic in its half plane of convergence. The convergence of the derived series $(f(n)/n^s)' = -f(n)\log(n)n^{-s}$ to $F'(s)$ follows from the fact that convergence is uniform in every compact subset of this half plane (see appendix A).

Since $\log n > 1$ for $n \geq 2$, it is clear that $\sum f(n)/n^s$ converges absolutely if $\sum (f(n)\log n)/n^s$ converges absolutely. To prove the other direction suppose that $\sum f(n)/n^s$ converges absolutely for $\sigma > \sigma_a$ and take s_1 with $\sigma_1 > \sigma_a$. It will follow that $\sum (f(n)\log n)/n^{s_1}$ converges absolutely since by taking σ_0 with $\sigma_1 > \sigma_0 > \sigma_a$ one has $\lim_{n \to \infty} n^{\sigma_0 - \sigma_1} \log n = 0$ and so there exists $N \in \mathbb{N}$ such that for $n \geq N$

$$\frac{\log n}{n^{\sigma_1 - \sigma_0}} \leq 1.$$

Therefore, for $n \geq N$

$$\left| \frac{f(n)\log n}{n^{s_1}} \right| = \frac{|f(n)|\log n}{n^{\sigma_1}} = \frac{|f(n)|}{n^{\sigma_0}} \frac{\log n}{n^{\sigma_1 - \sigma_0}} \leq \frac{|f(n)|}{n^{\sigma_0}}.$$

The result follows since σ_0 is in the half plane of absolute convergence of $\sum f(n)/n^s$. ■

1.2 General Properties

1.2.1 Theorem (Uniqueness)**.** *Let* $F(s) = \sum f(n)/n^s$ *and* $G(s) = \sum g(n)/n^s$ *be absolutely convergent for* $\sigma > \sigma_{FG}$*. If there exists a series* $\{s_k\}$ *in the half plane* $\sigma > \sigma_{FG}$ *with* $\lim_{k\to\infty} \sigma_k = \infty$ *such that* $F(s_k) = G(s_k)$ *for all* k*, then* $f(n) = g(n)$ *for all* n*.*

Proof. Define $H(s) = F(s) - G(s)$ and $h(n) = f(n) - g(n)$. By hypothesis $H(s_k) = 0$ for all k and by the convergence of F and G one has $H(s) = \sum h(n)/n^s$ for $\sigma > \sigma_{FG}$.

Suppose the the exists an n where $h(n) \neq 0$ and let N be the minimum number where this happens. It follows that

$$H(s) = \frac{h(N)}{N^s} + \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s}$$

and evaluating at s_k ($H(s_k) = 0$) and solving for $h(N)$ one obtains

$$h(N) = -N^{s_k} \sum_{n=N+1}^{\infty} \frac{h(n)}{n^{s_k}}.$$

Let $c \in \mathbb{R}$ be with $c > \sigma_{FG}$ and define $A = (N+1)^c \sum_{n=N+1}^{\infty} |h(n)|/n^c$ where $A < \infty$ by the absolute convergence of F and G. For every $\sigma_k \geq c$

$$\begin{aligned}
|h(N)| &\leq N^{\sigma_k} \sum_{n=N+1}^{\infty} \frac{|h(n)|}{n^{\sigma_k}} = N^{\sigma_k} \sum_{n=N+1}^{\infty} \frac{|h(n)|}{n^{\sigma_k - c} n^c} \\
&\leq \frac{N^{\sigma_k}}{(N+1)^{\sigma_k - c}} \sum_{n=N+1}^{\infty} \frac{|h(n)|}{n^c} \\
&= \left(\frac{N}{N+1}\right)^{\sigma_k} (N+1)^c \sum_{n=N+1}^{\infty} \frac{|h(n)|}{n^c} \\
&= A \left(\frac{N}{N+1}\right)^{\sigma_k}.
\end{aligned}$$

However, $\lim_{k\to\infty} \left(\frac{N}{N+1}\right)^{\sigma_k} = 0$ since $\lim_{k\to\infty} \sigma_k = \infty$. Therefore, $h(N) = 0$ which is a contradiction. ■

The following theorem shows one of the most useful properties of Dirichlet series in relation with number theory.

1.2.2 Theorem. *Let* $F(s) = \sum f(n)/n^s$ *and* $G(s) = \sum g(n)/n^s$ *be functions represented by Dirichlet series in* $\sigma > \sigma_F, \sigma_G$ *respectively. In the half plane* H_{FG} *where both series converge absolutely one has*

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{(f*g)(n)}{n^s} \tag{1.5}$$

where $f*g$ *is the* Dirichlet convolution of f and g *defined by*

$$(f*g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right) = \sum_{m_1 m_2 = n} f(m_1) g(m_2). \tag{1.6}$$

Moreover, the Dirichlet series (1.5) also converges absolutely in H_{FG}*.*

Proof. For any $s \in \mathbb{C}$ where both series converge absolutely we have

$$F(s)G(s) = \left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right) \left(\sum_{m=1}^{\infty} \frac{g(m)}{m^s}\right) = \sum_{n,m\geq 1} \frac{f(n)g(m)}{(nm)^s}$$

where the expression on the right is an absolutely convergent double series. Since any absolutely convergent double series can be rearranged in any simple series ([Apo67], pgs. 373-375), one can group the terms where $mn = k$ is constant to obtain

$$F(s)G(s) = \sum_{k=1}^{\infty} \frac{\sum_{nm=k} f(n)g(m)}{k^s} = \sum_{n=1}^{\infty} \frac{(f*g)(n)}{n^s}.$$

■

1.3 Representation as Infinite Products

The representation of Dirichlet series as infinite products follows from a theorem discovered by Euler in 1737. This theorem (stated and proved in what follows) is sometimes referred to as the analytic version of the fundamental theorem of arithmetic (representation in essentially a unique way of natural numbers as product of prime powers).

Throughout this section and what follows of this thesis, the notations $\sum_p a(p)$ and $\prod_p a(p)$ will be used for series and products over the set of prime numbers in ascending order. As an example, $\prod_p a(p)$ will mean $\prod_{n=1}^{\infty} a(p_n)$ where p_n is the n-th prime number. Before the theorem we introduce the following

1.3.1 Definition. Let $a : \mathbb{N} \to \mathbb{C}$ be different from zero for some n. $a(n)$ is said to be *multiplicative* if $a(nm) = a(n)a(m)$ whenever $(n, m) = 1$ (the greatest common divisor of n and m). $a(n)$ is said to be *completely multiplicative* if $a(nm) = a(n)a(m)$ for any $n, m \in \mathbb{N}$.

(Note that if $a(n)$ is multiplicative, then $a(1) = 1$.)

1.3.2 Theorem (Euler's Product Identity)**.** *Let $a : \mathbb{N} \to \mathbb{C}$ and suppose that $\sum a(n)$ converges absolutely. If $a(n)$ is multiplicative, then*

$$\sum_{n=1}^{\infty} a(n) = \prod_p \left(1 + \sum_{n=1}^{\infty} a(p^n)\right). \tag{1.7}$$

If $a(n)$ is completely multiplicative,

$$\sum_{n=1}^{\infty} a(n) = \prod_p \left(\frac{1}{1 - a(p)}\right). \tag{1.8}$$

Proof. Given that $\sum a(n)$ is absolutely convergent, the series $\sum_{n=1}^{\infty} a(p^n)$ is absolutely convergent for each prime p since this is a subseries of $\sum a(n)$. Define

$$P_k = \prod_{p \le k} \left(1 + \sum_{n=1}^{\infty} a(p^n)\right)$$

where the product is taken over all primes less that or equal to k. P_k is a finite product of absolutely convergent series. Applying inductively the the theorems on products of absolutely convergent series as double series and rearranging these as simple series it follows that P_k can be expressed as an absolutely convergent simple series whose terms (with the exception of 1) are of the form

$$a(p_1^{\alpha_1})a(p_2^{\alpha_2}) \dots a(p_t^{\alpha_t}) = a(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t})$$

with $p_i \le k$ different primes and $\alpha_i > 0$ for $i = 1, \dots, t$. The last equality is due to the fact that $a(n)$ is multiplicative.

By the fundamental theorem of arithmetic there will be only one term $a(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t})$ for each choice of primes smaller than k and exponents $\alpha_i > 0$. Taking $A_k \subset \mathbb{N}$ as the set of natural numbers whose prime decomposition only contains primes smaller than k (including 1) it follows that

$$P_k = \sum_{n \in A_k} a(n).$$

Therefore,

$$\left|\sum_{n=1}^{\infty} a(n) - P_k\right| = \left|\sum_{n \notin A_k} a(n)\right| \le \sum_{n \notin A_k} |a(n)| \le \sum_{n \ge k} |a(n)|$$

since $n \ge k$ if $n \notin A_k$. By the absolute convergence of $\sum a(n)$ one has that $\lim_{k\to\infty} \sum_{n \ge k} |a(n)| = 0$, and so

$$\sum_{n=1}^{\infty} a(n) = \lim_{k\to\infty} P_k = \prod_p \left(1 + \sum_{n=1}^{\infty} a(p^n)\right).$$

If $a(n)$ is completely multiplicative, $|a(p)| < 1$ for every prime p since otherwise $\sum_{n=1}^{\infty} |a(p^n)| = \sum_{n=1}^{\infty} |a(p)|^n$ would diverge contradicting the absolute convergence of $\sum a(n)$. Therefore, For every prime p

$$1 + \sum_{n=1}^{\infty} a(p^n) = \sum_{n=0}^{\infty} [a(p)]^n = \frac{1}{1 - a(p)}.$$

■

The representation of these series as products allows us to prove the following non trivial corollary based on the properties of infinite products included in the appendix C.

1.3.3 Corollary. *If $\sum a(n)$ converges absolutely and $a(n)$ is completely multiplicative then $\sum a(n) \neq 0$.*

Proof. Given (1.8), one only has to prove that the product is different from zero. Since $\sum_p a(p)$ converges absolutely and $a(p) \neq -1$ for every prime (by arguments in the previous theorem), it follows from C.4 in the appendix that $\prod_p [1 - a(p)]$ converges and is different from zero. Therefore, by theorem C.2 one has

$$\sum_{n=1}^{\infty} a(n) = \prod_p \left(\frac{1}{1 - a(p)} \right) = \frac{1}{\prod_p [1 - a(p)]} \neq 0.$$

■

1.3.4 Corollary (Representation as Products)**.** *Let $\sum f(n)/n^s$ be an absolutely convergent Dirichlet series for $\sigma > \sigma_a$ and suppose that $f(n)$ is multiplicative. Then, in the half plane of absolute convergence*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \sum_{n=1}^{\infty} \frac{f(p^n)}{p^{ns}} \right) \qquad \sigma > \sigma_a.$$

If $f(n)$ is completely multiplicative, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(\frac{1}{1 - f(p) p^{-s}} \right) \neq 0$$

for every $s \in \mathbb{C}$ with $\sigma > \sigma_a$.

Proof. Take $a(n) = f(n)/n^s$ and use 1.3.2 and 1.3.3 taking into account that $a(n)$ is multiplicative if $f(n)$ is multiplicative and that $a(n)$ is completely multiplicative if $f(n)$ is completely multiplicative since

$$a(nm) = \frac{f(nm)}{(nm)^s} = \frac{f(n)f(m)}{n^s m^s} = \frac{f(n)}{n^s} \frac{f(m)}{m^s} = a(n)a(m).$$

■

1.4 The Multiplicative inverse of a Dirichlet Series

From corollary 1.3.4 it follows that Dirichlet series are different from zero in their half plane of absolute convergence if their coefficients are completely multiplicative. In this section a Dirichlet series for $(\sum f(n)/n^s)^{-1}$ will be calculated in this half plane.

The sought for series will be a consequence of a more general theorem where a series for $(\sum a(n))^{-1}$ is computed when $\sum a(n)$ is absolutely convergent and $a(n)$ is completely multiplicative. Under these hypothesis,

$$\frac{1}{\sum a(n)} = \prod_p [1 - a(p)] \tag{1.9}$$

by 1.3.2 and 1.3.3.

If one wishes to express $(\sum a(n))^{-1}$ as a series it is only natural to use (1.9) to find this series.

Define $P_k = \prod_{p \leq k} [1 - a(p)]$. If one expands this finite product the result will be a sum with terms of the form

$$(-1)^r a(p_1) a(p_2) \ldots a(p_r) = (-1)^r a(p_1 p_2 \ldots p_r)$$

with $p_1, \ldots, p_r$ different primes less than k.

If we define again $A_k \subset \mathbb{N}$ as the integers whose prime divisors are all less than or equal than k one is led naturally to define $\mu : \mathbb{N} \to \mathbb{R}$ by

$$\mu(n) = \begin{cases} 1 & if \quad n = 1 \\ (-1)^k & if \quad \text{n is the product of k different primes} \\ 0 & \quad\text{otherwise} \end{cases} \tag{1.10}$$

(this function is known as the Möbius μ function) to write $P_k = \sum_{n \in A_k} \mu(n)a(n)$ from which one conjectures that

$$\frac{1}{\sum a(n)} = \sum_{n=1}^{\infty} \mu(n)a(n).$$

The last equation is actually true and is proved in the next theorem.

1.4.1 Theorem. *Let $a(n)$ be completely multiplicative and suppose that $\sum a(n)$ converges absolutely. Then,*

$$\left(\sum_{n=1}^{\infty} a(n)\right)^{-1} = \sum_{n=1}^{\infty} \mu(n)a(n)$$

where μ is the Möbius μ function defined by (1.10). The series on the right converges absolutely .

Proof. $\sum \mu(n)a(n)$ converges absolutely since $|\mu(n)a(n)| \leq |a(n)|$. Using the same notation as in the previous discussion one has

$$\left|\sum \mu(n)a(n) - P_k\right| \leq \sum_{n \notin A_k} |\mu(n)a(n)| \leq \sum_{n \geq k} |\mu(n)a(n)|\,.$$

Now, since the term on the right tends to zero when $k \to \infty$,

$$\sum_{n=1}^{\infty} \mu(n)a(n) = \lim_{k \to \infty} P_k = \left(\sum_{n=1}^{\infty} a(n)\right)^{-1}.$$

■

1.4.2 Corollary (The Multiplicative inverse of a Dirichlet Series)**.** *Let $\sum f(n)/n^s$ be a Dirichlet series with non empty half plane of convergence and suppose that $f(n)$ is completely multiplicative. Then, in the half plane of absolute convergence $\sigma > \sigma_a$ it is true that*

$$\left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)f(n)}{n^s}$$

and the Dirichlet series on the right also converges absolutely for $\sigma > \sigma_a$.

1.5 The Logarithm of a Dirichlet Series

Given that the logarithmic function is not single valued on the complex numbers, the logarithm of a Dirichlet series cannot be expressed as a Dirichlet series since the later are single valued. We will find however a Dirichlet series which serves as one of the logarithms of any given Dirichlet series.

The first step will be to find a Dirichlet series for $F'(s)/F(s)$ under some restrictions to F and to find the sought for series from this one. The following function, and accompanying property will result extremely useful.

1.5.1 Definition. The *von Mangoldt* function $\Lambda : \mathbb{N} \to \mathbb{R}$ is defined by:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with } p \text{ prime and } k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

1.5.2 Lemma. *If $f(n)$ is completely multiplicative and g is defined as $g(n) = f(n)\Lambda(n)$ ($g = f\Lambda$), then*

$$(g * f)(n) = f(n) \log n.$$

Proof. Since $f(n)$ is completely multiplicative,

$$(g * f)(n) = \sum_{d|n} f(d)\Lambda(d)f\left(\frac{n}{d}\right) = f(n)\sum_{d|n}\Lambda(d)$$

and so it suffices to prove that

$$\sum_{d|n}\Lambda(d) = \log n. \tag{1.11}$$

If $n = 1$ the equality is satisfied so assume $n > 1$ and write $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$. Evaluating the left had side of (1.11) one gets

$$\sum_{d|n}\Lambda(d) = \sum_{i=1}^{k}\sum_{j=1}^{\alpha_i}\Lambda(p_i^j) = \sum_{i=1}^{k}\alpha_i \log p_i = \log n.$$

since for every divisor d of n, $\Lambda(n) \neq 0$ if and only if d is a power of a prime. ∎

1.5.3 Theorem. *Let $F(s) = \sum f(n)/n^s$ be a Dirichlet series with $f(n)$ completely multiplicative. For any s in the half plane of absolute convergence of $F(s)$,*

$$\frac{F'(s)}{F(s)} = -\sum_{n=1}^{\infty}\frac{f(n)\Lambda(n)}{n^s}$$

where the Dirichlet series on the right also converges absolutely in the same half plane.

Proof. Let H_F be the half plane of absolute convergence of F and define G by

$$G(s) = -\sum_{n=1}^{\infty}\frac{f(n)\Lambda(n)}{n^s}.$$

It suffices to prove that $F'/F = G$ or equivalently that $F' = GF$.

Now, G converges absolutely in H_F since

$$F'(s) = -\sum_{n=1}^{\infty}\frac{f(n)\log(n)}{n^s}$$

converges absolutely in H_F and $|f(n)\Lambda(n)/n^s| \leq |f(n)\log(n)/n^s|$ for all n. Therefore, by the theorem 1.2.2 on products of absolutely convergent Dirichlet series,

$$G(s)F(s) = -\sum_{n=1}^{\infty}\frac{(f\Lambda * f)(n)}{n^s}.$$

And so, by lema lema 1.5.2, $F' = GF$ in H_F since $f\Lambda * f = f\log$. ∎

1.5.4 Lemma. *If $\sum f(n)/n^s$ has a non empty half plane of absolute convergence,*

$$\lim_{\sigma\to\infty}\sum_{n=2}^{\infty}\frac{f(n)}{n^s} = 0.$$

Proof. Choose a real c in the half plane of absolute convergence and define

$$A = \sum_{n=2}^{\infty}\frac{|f(n)|}{n^c}.$$

For every $s \in \mathbb{C}$ with $\sigma \geq c$ one has

$$\left|\sum_{n=2}^{\infty}\frac{f(n)}{n^s}\right| \leq \sum_{n=2}^{\infty}\frac{|f(n)|}{n^\sigma} = \sum_{n=2}^{\infty}\frac{|f(n)|}{n^c n^{\sigma-c}} \leq \frac{A}{2^{\sigma-c}}.$$

The theorem follows from the fact that $\lim_{\sigma\to\infty} A/2^{\sigma-c} = 0$. ∎

1.5.5 Theorem (Logarithm of a Dirichlet Series)**.** *Let $F(s) = \sum f(n)/n^s$ be a Dirichlet series $f(n)$ completely multiplicative. For any s in the half plane of absolute convergence one has*

$$F(s) = e^{G(s)}$$

where

$$G(s) = \sum_{n=2}^{\infty} \frac{f(n)\Lambda(n)}{n^s \log n}.$$

Proof. Let H_F be the half plane of absolute convergence of $F(s)$. Since $F(s) \neq 0$ in H_F (Corollary 1.4.2) and F is analytic in H_F, there exists an analytic function $G(s)$ defined on H_F which satisfies (look for example in [Con73], p. 87)

$$F(s) = e^{G(s)} \qquad s \in H_F.$$

Differentiating this equation one obtains $F'(s) = G'(s)e^{G(s)} = G'(s)F(s)$. Therefore,

$$G'(s) = \frac{F'(s)}{F(s)} = -\sum_{n=2}^{\infty} \frac{f(n)\Lambda(n)}{n^s} \tag{1.12}$$

by theorem 1.5.3. Integrating term by term which is possible since convergence is locally uniform in H_F, one obtains

$$G(s) = C + \sum_{n=2}^{\infty} \frac{f(n)\Lambda(n)}{n^s \log n}.$$

Lema 1.5.4 is used to find C. Given that $F(s) = e^{G(s)}$ and that the exponential is continuous,

$$\lim_{\sigma\to\infty} F(s) = e^{\lim_{\sigma\to\infty} G(s)}$$

which by the lema translates into $f(1) = e^C$. However, since $f(n)$ is multiplicative and different form zero for at least some n, $f(1) = 1$ and so C is a complex logarithm of 1. In particular one can take $C = 0$. ■

The Dirichlet series for $G(s)$ can also be obtained from the product representation of $F(s)$ (1.3.4). This alternate way gives some motivation for the definition of the $\Lambda(n)$ function but uses more delicate arguments. One basically uses the fact that

$$\sum_p -\log(1 - f(p)/p^s)$$

converges where log stands for the principal branch of the logarithm (argument between $-\pi$ and π). This is proved using the Taylor series of $\log(1-z)$ (the principal branch) convergent for $|z| < 1$. Note that $|f(p)/p^s| < 1$ for every prime p under the hypothesis of the theorem. Using this fact and the continuity of the exponential function, the series is a logarithm of $F(s)$. After some algebraic manipulation one manages to rewrite this series as a Dirichlet series (Look for example in [Apo76] or [Jam03]).

1.6 An Application: Möbius Inversion

On section 1.4 it was seen how Euler's product formula motivates the definition of the Möbius μ function. In this section, somehow outside the main theme of this work, we will se how the use of Dirichlet series leads naturally to the number theoretic formula known as the Möbius inversion formula from classical number theory.

Suppose one has complex sequences $\{a_n\}_{n\in\mathbb{N}}$,$\{b_n\}_{n\in\mathbb{N}}$ satisfying

$$a_n = \sum_{d|n} b_d$$

and one wishes to find an expression for b_n in terms of the sequence $\{a_n\}_{n\in\mathbb{N}}$.

Theorem 1.2.2 on products of Dirichlet series suggests considering the Dirichlet series $\sum 1/n^s$ (convergence issues will be ignored since they will be studied in the next chapter) since if $\sum a_n/n^s$,$\sum b_n/n^s$ and $\sum 1/n^s$ converge for some $s \in \mathbb{C}$ then by the theorem one can write

$$\sum \frac{a_n}{n^s} = \left(\sum \frac{b_n}{n^s}\right)\left(\sum \frac{1}{n^s}\right)$$

for every $s \in \mathbb{C}$ where all series converge absolutely.

By 1.4.2 one also has that

$$\frac{1}{\sum 1/n^s} = \sum \frac{\mu(n)}{n^s},$$

and so,

$$\sum \frac{b_n}{n^s} = \left(\sum \frac{a_n}{n^s}\right)\left(\sum \frac{\mu(n)}{n^s}\right) = \sum \frac{(\mu * a)(n)}{n^s}.$$

Therefore, by the theorem on uniqueness of coefficients 1.2.1 one will have

$$b_n = \sum_{d|n} a_d \mu\left(\frac{n}{d}\right).$$

This last result is precisely the Möbius inversion formula and is widely used in classical number theory. It is worthwhile to point out that it is true without the requirement on the convergence of the Dirichlet series used in the previous argument and its proof is elementary but ill-motivated. In this context, the hypothesis on the convergence of the series can be avoided by defining $a'_n = a_n$ if $n < N$ and zero otherwise (likewise with b_n) and proving the formula inductively.

Chapter 2

Riemann's ζ function

2.1 Definition of the ζ function and general properties

The Riemann ζ function is defined as the series $\sum_{n=1}^{\infty} 1/n^s$ wherever it converges. The next theorem describes the region where it converges.

2.1.1 Theorem. *The Dirichlet Series $\sum 1/n^s$ converges absolutely if $\sigma > 1$ and diverges if $\sigma < 1$. In this way, following the notation of theorem 1.1.2, $\sigma_a = \sigma_c = 1$.*

Proof. One has $|1/n^s| = 1/n^\sigma$. Absolute convergence for $\sigma > 1$ follows from the convergence of the integral $\int_1^\infty 1/x^\sigma dx$. The statement follows from 1.1.2 since $\sum 1/n$ diverges. ■

2.1.2 Definition. Riemann's ζ function is defined for Re $s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In what follows, the consequences from the theory on general Dirichlet Series to the Riemann ζ function will be given. In all of them s will be assumed to have real part $\sigma > 1$.

2.1.3 Theorem. *The ζ function is analytic and $\zeta'(s)$ can be expressed as an absolutely convergent Dirichlet Series for $\sigma > 1$ by*

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}.$$

2.1.4 Theorem. *ζ admits the following representation as an infinite product over the prime numbers*

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_p \frac{p^s}{p^s - 1}.$$

2.1.5 Theorem. *$\zeta(s) \neq 0$.*

Proof. By the product representation. ■

2.1.6 Theorem. *The function $1/\zeta(s)$ is analytic and can be expressed as an absolutely convergent Dirichlet Series for $\sigma > 1$ by:*

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where μ is the Möbius μ function defined in (1.10).

2.1.7 Theorem. *ζ'/ζ can be expressed as an absolutely convergent Dirichlet Series for $\sigma > 1$ by*

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

where $\Lambda(n)$ is the von Mangoldt *function defined in 1.5.1.*

2.1.8 Definition. $\log\zeta(s)$ is defined for Re $s > 1$ by

$$\log\zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)/\log n}{n^s}.$$

2.1.9 Theorem. *$\log\zeta(s)$ is a logarithm of $\zeta(s)$ ($e^{\log\zeta(s)} = \zeta(s)$), it is real when s is real, and the Dirichlet Series which represents is absolutely convergent where it is defined (Re $s > 1$).*

2.2 Relation of the ζ with Number Theory

In this section, the relations between the ζ function and Number Theory will be explored. The first mathematician to note the connections between ζ and the integers was Euler. His studies concerning the ζ function led him to results such as 1.3.2 and an analytic proof in the infinitude of prime numbers, which will be shown in what follows. The content of this section will not be used in other parts of this document.

2.2.1 Lemma. *For real σ with $\sigma > 1$, $\zeta(\sigma) \geq 1/(\sigma - 1)$.*

Proof. By B.3,

$$\sum_{n=1}^{N} \frac{1}{n^\sigma} > \int_1^N \frac{1}{x^\sigma} dx = \frac{1 - N^{1-\sigma}}{\sigma - 1}.$$

The result follows making $N \to \infty$. ■

2.2.2 Theorem (Modification of Euler's Proof)**.** *There exist an infinite amount of Prime numbers.*

Proof. Take the equality $\zeta(\sigma) = \prod_p \left(1 - p^{-\sigma}\right)^{-1}$ with real $\sigma > 1$ and make $\sigma \to 1^+$. The existence of only a finite number of primes would imply that the product converges to the rational number $\prod_p \left(1 - p^{-1}\right)^{-1}$ but this contradicts the fact that $\zeta(\sigma)$ is unbounded when $\sigma \to 1^+$ by the previous lema. ■

Even if it seems like a very long road to prove a theorem that has proofs as simple as the one given in Euclid's Elements, its importance lies in the fact that it was the first proof in which the tools of analysis were used to prove facts about integers. Moreover, the proof establishes a link between the Fundamental Theorem of Arithmetic and the existence of infinite primes since the representation of ζ as an infinite product from 2.1.4 is obtained almost as a consequence of Fundamental Theorem of Arithmetic.

One can also prove the divergence of the series $\sum_p 1/p$ following the previous line of proof, as was first done by Euler. I will not develop on such a proof, but the interested reader can take a look of a proof of this fact using the ζ function in [Gro84]. Euler's original proof of theorem 2.2.2 may be found in [Dia96].

There are certain very intriguing equalities concerning the ζ function and common arithmetic functions in Elementary Number Theory. Some of these equalities will be shown in what follows. Definitions of the functions concerned are included as reference.

2.2.3 Definition. $\phi(n)$ is the number of integers $\leq n$ which are relatively prime with n.

ϕ is known as the *Euler ϕ function.* The following property of ϕ can be found on any text of Elementary Number Theory. (eg, [Rub99], pg. 74).

2.2.4 Theorem. *$\sum_{d|n} \phi(d) = n$ for all n.*

2.2.5 Theorem. *$\sum \phi(n)/n^s$ converges absolutely for Re $s > 2$ and*

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} = \prod_p \frac{1 - p^{-s}}{1 - p^{1-s}}.$$

Proof. Absolute convergence follows from the fact that $\phi(n) \leq n$, therefore $\sum \phi(n)/n^s \leq \sum 1/n^{s-1} = \zeta(s-1)$. For the first equality note that by theorems 2.2.4 and 1.2.2,

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\sum_{d|n} \phi(d)}{n^s} = \sum_{n=1}^{\infty} \frac{n}{n^s} = \zeta(s-1).$$

The equality follows since $\zeta(s) \neq 0$. The second equality is a consequence of 2.1.4. ■

2.2.6 Definition. The function σ_α with $\alpha \in \mathbb{C}$ is defined by

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha.$$

σ_0 is usually denoted by τ. Note that $\tau(n)$ is the number of divisors of n. σ_1, the sum of the divisors of n is usually denoted by σ.

2.2.7 Theorem. *$\sum \sigma_\alpha(n)/n^s$ converges absolutely for Re $s > max\{1, 1+$Re $\alpha\}$ and*

$$\sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^s} = \zeta(s)\zeta(s-\alpha).$$

Proof. Take $f(n) = n^\alpha$. Since $\sum f(n)/n^s = \zeta(s-\alpha)$, $\sum f(n)/n^s$ converges absolutely for Re $s > 1+$Re α. This implies that

$$\zeta(s)\zeta(s-\alpha) = \sum_{n=1}^{\infty} \frac{\sum_{d|n} f(d)}{n^s} = \sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^s}$$

converges absolutely for Re $s > max\{1, 1+$Re $\alpha\}$. ■

The previous equalities can be used to prove identities about convolution in a very straight forward manner. For example, the identity

$$\sum_{d|n} \phi(d)\tau(n/d) = \sigma(n)$$

is a direct consequence of

$$\frac{\zeta(s-1)}{\zeta(s)}\zeta^2(s) = \zeta(s-1)\zeta(s)$$

by the theorems on products of Dirichlet Series and uniqueness of coefficients since $\sum_{d|n} \phi(d)\tau(n/d) = (\phi * \tau)(n)$.

The following result if of great importance since from it one can deduce the equation which Riemann used in his revolutionary (and only) article on Number Theory (See [Gro84] and [Rie59]).

2.2.8 Theorem. *Denote by $\pi(x)$ the number of primes which are $\leq x$ and consider $\sigma \in \mathbb{R}$ with $\sigma > 1$. Then,*

$$\log \zeta(\sigma) = \sigma \int_2^\infty \frac{\pi(x)}{x(x^\sigma - 1)} dx.$$

Proof. Given the equality $\zeta(\sigma) = \prod_p (1-p^{-\sigma})^{-1}$,

$$\begin{aligned}\log \zeta(\sigma) &= -\sum_p \log(1-p^{-\sigma}) \\ &= -\sum\nolimits_{n=2}^{\infty} [\pi(n) - \pi(n-1)] \log(1-n^{-\sigma})\end{aligned}$$

were the last equality follows from the fact that $\pi(n) - \pi(n-1)$ is the characteristic function of prime numbers.

Using Abel's Identity B.1 with $a(n) = \pi(n) - \pi(n-1)$, $f(x) = \log(1-x^{-\sigma})$, $y = 2$ and $x = N > 2$ one obtains

$$\begin{aligned}\sum_{n=3}^{N} a(n)\log(1-n^{-\sigma}) =&\pi(N)\log(1-N^{-\sigma}) - \pi(2)\log(1-2^{-\sigma}) \\ &- \sigma \int_2^N \frac{\pi(x)}{x(x^\sigma - 1)} dx\end{aligned}$$

since $A(x) = \sum_{n \leq x} [\pi(n) - \pi(n-1)] = \pi(x)$ and so,

$$\sum_{n=2}^{N} [\pi(n) - \pi(n-1)] \log(1-p^{-\sigma}) = \pi(N)\log(1-N^{-\sigma}) - \sigma \int_2^N \frac{\pi(x)}{x(x^\sigma - 1)} dx.$$

However, since $\pi(n) \leq n$, $\lim_{N\to\infty} \pi(N)\log(1-N^{-\sigma}) = 0$ and the equality follows by taking $N \to \infty$. ■

It can be proved that the previous equality also holds replacing σ by $s \in \mathbb{C}$ with Re $s > 1$ and taking $\log \zeta(s)$ as the function defined in 2.1.8.

2.3 Analytic Continuation of ζ

This section relies heavily on the properties (analyticity in particular) of functions defined by a certain type of integral known as Dirichlet Integrals. The following theorem is proved in the Appendix D. Note the similarity with 1.1.4.

2.3.1 Theorem (Appendix D.3). *Let $N \in \mathbb{N}$, $N \neq 0$ and $f : [N, \infty) \to \mathbb{C}$ continuous except possibly at $\mathbb{N} \cap [N, \infty)$ where however, right and left limits exist. If there exist $M, \alpha \in \mathbb{R}$ such that $|f(x)| \leq Mx^{\alpha}$, then*

$$\int_N^\infty \frac{f(x)}{x^{s+1}} dx$$

converges for $s \in \mathbb{C}$ with $\sigma > \alpha$ and if one defines $I(s)$ as the value of this integral, then $I(s)$ is analytic in Re $s > \alpha$ and its derivative is given by

$$I'(s) = -\int_N^\infty \frac{f(x) \log x}{x^{s+1}} dx.$$

The next theorem shows us a way to express ζ in terms of integrals. Its main use is that the expression is analytic in a region larger than Re $s > 1$. This allows to extend the definition of the ζ function.

2.3.2 Theorem. *The following equality is valid for all $N \in \mathbb{N}$*

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s \int_N^\infty \frac{x - [x]}{x^{s+1}} dx \qquad \text{Re } s > 1$$

and the integral on the right is analytic (as a function of s) in the half-plane Re $s > 0$.

Proof. Taking $f(x) = 1/x^s$ in Euler's Summation Formula B.2 with $y = N$, $x = M > N$, N and M integers,

$$\sum_{n=N+1}^{M} \frac{1}{n^s} = \int_N^M \frac{1}{x^s} dx - s \int_N^M \frac{x - [x]}{x^{s+1}} dx. \tag{2.1}$$

Take s with $\sigma > 1$ and make $M \to \infty$ in (2.1). All the terms converge since the sum is the tail of the ζ function and the integrals satisfy the hypothesis in 2.3.1. Therefore,

$$\sum_{n=N+1}^{\infty} \frac{1}{n^s} = \int_N^\infty \frac{1}{x^s} dx - s \int_N^\infty \frac{x - [x]}{x^{s+1}} dx.$$

Evaluating the integral on the left and adding $\sum_{n=1}^{N} 1/n^s$ on both sides of the equation one obtains the stated equality. The fact that

$$-s \int_N^\infty \frac{x - [x]}{x^{s+1}} dx$$

is analytic for $\sigma > 0$ is a consequence of 2.3.1 taking $f(x) = x - [x] \leq 1$ and $\alpha = 0$. ■

The expression obtained using $N = 1$ in 2.3.2 is particularly simple and motivates the following.

2.3.3 Definition. $\zeta(s)$ is defined for Re $s > 0$, $s \neq 1$ by

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \frac{x - [x]}{x^{s+1}} dx.$$

From now onwards, whenever we mention the function ζ we will be referring to the function in the previous definition. It is clear that by theorem 2.3.2, this function coincides with $\sum 1/n^s$ if $\sigma > 1$. It should be noticed however that the theorems proved before 2.3.2 are only valid for $\sigma > 1$.

In particular, *Riemann's Hypothesis*, still without proof, states that the only zeros of ζ with positive real part have real part $\sigma = 1/2$ (this did not make sense before the analytic continuation). Note that 2.1.5 guarantees that if any zeros exist, these must have real part $\sigma \leq 1$. An essential part of the proof of the Prime Number Theorem that will be shown in this document will depend essentially on the behaviour of the zeros with $\sigma = 1$. It will be shown in the next section that ζ

has no zeros with real part $\sigma = 1$ by the use of some quite technical but very ingenious arguments. The behaviour of the zeros of ζ has eluded the efforts of the most eminent mathematicians. It seems incredible that no mathematician has even been able to prove the existence of a region of the form Re $s \geq c$ with $c < 1$ where ζ has no zeros.

As a last comment about Riemann's Hypothesis, the restriction *zeros with real part* comes from the fact that the ζ function can be continued analytically to the whole complex plane with the use o a functional equation initially discovered by Euler in the real case and finally proved by Riemann. This continuation has zeros in the negative even integers and it is know that they are the only zeros with negative real part. This continuation however will be of no use in the present document, but the interested reader may look in [Apo76], [Gro84] or [Jam03] for the details.

2.3.4 Theorem. *ζ is analytic in Re $s > 0$ except at the point $s = 1$ where it has a simple pole with residue* 1.

Proof. Simply note that

$$\zeta(s) - \frac{1}{s-1} = 1 - s\int_1^\infty \frac{x - [x]}{x^{s+1}}dx$$

and the expression on the right is analytic for $s > 0$ by the previous theorem. ∎

2.3.5 Corollary. $\lim_{s\to 1}(s-1)\zeta(s) = 1$

Now we will prove that the identities in 2.3.2 are still valid for our new ζ function. They are of course still true if Re $s > 1$ and by a simple application of a theorem on uniqueness of analytic continuation one can conclude that it is still valid for Re $s > 0$. However, a direct proof can be given:

2.3.6 Theorem. *For Re $s > 0$ and for every $N \in \mathbb{N}$ with $N > 1$, $\zeta(s)$ can be expressed as*

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^\infty \frac{x-[x]}{x^{s+1}}dx.$$

Proof. By definition,

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{x-[x]}{x^{s+1}}dx. \tag{2.2}$$

Taking $f(x) = 1/x^s$ in Euler's Summation Formula B.2 with $y = 1$ and $x = N$ one obtains

$$\sum_{n=2}^{N} \frac{1}{n^s} = \int_1^N \frac{1}{x^s}dx - s\int_1^N \frac{x-[x]}{x^{s+1}}dx. \tag{2.3}$$

Evaluating the first integral and adding 1 in (2.3) it follows that

$$\sum_{n=1}^{N} \frac{1}{n^s} = 1 + \frac{1}{s-1} - \frac{N^{1-s}}{s-1} - s\int_1^N \frac{x-[x]}{x^{s+1}}dx. \tag{2.4}$$

Finally, subtracting (2.4) from (2.2) one obtains the desired identity. ∎

2.3.7 Theorem. *For Re $s > 0$ and $s \neq 1$,*

$$\zeta'(s) = -\sum_{n=1}^{N} \frac{\log n}{n^s} + s\int_N^\infty \frac{(x-[x])\log x}{x^{s+1}}dx - \int_N^\infty \frac{x-[x]}{x^{s+1}}dx$$
$$- \frac{N^{1-s}\log N}{s-1} - \frac{N^{1-s}}{(s-1)^2}.$$

Proof. The usual rules of differentiation and theorem 2.3.1 for the derivative of the integral. ∎

2.4 Proof of $\zeta(1+it) \neq 0$ for $t \neq 0$

2.4.1 Lemma. *Let $F(s) = \sum f(n)/n^s$ be an absolutely convergent Dirichlet Series for $\sigma > \sigma_a$ with f real and positive. If $s = \sigma + it$ with $\sigma > \sigma_a$, then*

$$3F(\sigma) + 4Re\ F(\sigma + it) + Re\ F(\sigma + 2it) \geq 0.$$

Proof. One has

$$3F(\sigma) + 4F(\sigma + it) + F(\sigma + 2it) = \sum_{n=1}^{\infty} \frac{f(n)}{n^\sigma}(3 + 4n^{-it} + n^{-2it}).$$

Taking real parts and noticing that Re $n^{-ai} =$ Re $e^{-ia \log n} = \cos(a \log n)$ it follows that

$$3F(\sigma) + 4\text{Re } F(\sigma + it) + \text{Re } F(\sigma + 2it) = \sum_{n=1}^{\infty} \frac{f(n)}{n^\sigma}(3 + 4\cos(t \log n) + \cos(2t \log n)). \tag{2.5}$$

However, for every $\alpha \in \mathbb{R}$,

$$3 + 4\cos(\alpha) + \cos(2\alpha) = 2 + 4\cos(\alpha) + 2\cos^2(\alpha) = 2(1 + \cos(\alpha))^2 \geq 0.$$

Therefore, the Dirichlet Series in (2.5) is ≥ 0 since f is positive. ■

2.4.2 Corollary. *If $\sigma > 1$ then*

$$\zeta^3(\sigma)\,|\zeta(\sigma + it)|^4\,|\zeta(\sigma + 2it)| \geq 1.$$

Proof. Consider the function $\log \zeta(s)$ defined by 2.1.8. Since $\log |\zeta(s)| =$ Re $\log \zeta(s)$, $\log \zeta(s)$ is real when s is real, and $\Lambda(n)$ is positive. It follows by lema 2.4.1 that

$$\begin{aligned} \zeta^3(\sigma)\,|\zeta(\sigma + it)|^4\,|\zeta(\sigma + 2it)| &= e^{3\log\zeta(\sigma) + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+i2t)|} \\ &\geq e^0 = 1. \end{aligned}$$

■

The following theorem, proved independently by Hadamard and de la Valle Poussin, was the first result on the zeros of ζ in the region $0 \leq$ Re $s \leq 1$. Using this result, they also independently gave the first proofs of the Prime Number Theorem. The ideas present in the present exposition are simplifications made by Mertens and de la Valle Poussin of an argument of Hadamard.

2.4.3 Theorem. $\zeta(1 + it) \neq 0$ *for* $t \neq 0$.

Proof. By the previous corollary, if $\sigma > 1$ then

$$\zeta^3(\sigma)\,|\zeta(\sigma + it)|^4\,|\zeta(\sigma + 2it)| \geq 1.$$

multiplying and dividing by $(\sigma - 1)^4$ one obtains,

$$[(\sigma - 1)\zeta(\sigma)]^3 \left|\frac{\zeta(\sigma + it)}{\sigma - 1}\right|^4 |(\sigma - 1)\zeta(\sigma + i2t)| \geq 1. \tag{2.6}$$

If one assumes the existence of a $t \neq 0$ such that $\zeta(1 + it) = 0$ one reaches a contradiction with the previous inequality taking $\sigma \to 1^+$ since the term $[(\sigma - 1)\zeta(\sigma)]^3 \to 1$ by 2.3.5, the term $|(\sigma - 1)\zeta(\sigma + i2t)| \to 0$ because $\zeta(\sigma + i2t) \to \zeta(1 + i2t)$ by continuity, and the limit

$$\lim_{\sigma \to 1^+} \frac{\zeta(\sigma + it)}{\sigma - 1}$$

exists since under the hypothesis $\zeta(1 + it) = 0$, and so it is simply $\zeta'(1 + it)$ which is finite since ζ is analytic. In this manner,

$$\lim_{\sigma \to 1^+} [(\sigma - 1)\zeta(\sigma)]^3 \left|\frac{\zeta(\sigma + it)}{\sigma - 1}\right|^4 |(\sigma - 1)\zeta(\sigma + i2t)| = 0$$

which contradicts (2.6). ■

For the chain of ideas that led Hadamard to these arguments look in [Gol73].

Chapter 3

The Prime Number Theorem

The Prime Number Theorem (PNT) states that:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1 \tag{3.1}$$

where $\pi(x)$ is the number of primes $\leq x$. The asymptotic equality in (3.1) is usually expressed as $\pi(x) \sim x/\log(x)$ adopting the convention that $f(x) \sim g(x)$ means $\lim_{x\to\infty} f(x)/g(x) = 1$.

Given the analytic properties of Dirichlet Series, and the close relationship which these have with the multiplicative structure of the integers, Dirichlet series become a valuable tool for the study of multiplicative functions since the behaviour of the series can shed light on the nature of its coefficients. The purpose of this chapter is to show one of the ways in which Dirichlet Series allow one to understand the nature of its coefficients. This will eventually result in a proof of the PNT. In essence, the arguments which follow come from [Jam03] which in turn is a generalization of one of the various proofs of the PNT using Complex Analysis. Even though this exposition somewhat obscures the proof of the PNT, it shows that a more general result is valid.

3.1 Strategy and Main Ideas of the Proof

As was already said, the main idea is to take the Dirichlet Series $F(s) = \sum f(n)/n^s$ and obtain information about the coefficients from the analytical properties of F. Specifically, an asymptotic identity involving the partial sums $S_f(x) = \sum_{n\leq x} f(n)$ will be obtained. In the case of the PNT the idea is of course to take $f(n)$ bearing some relation with prime numbers.

A first choice for f would of course be the characteristic function of prime numbers in which case $S_f(x) = \pi(x)$. However, this f is not multiplicative and so various of the theorems proved about Dirichlet Series are not applicable. It will be noticed later that they turn out to be essential in the proof of the PNT. As far as I am aware of, there is no analytical proof of the PNT using $\pi(x)$ directly.

Even thought the relation between Dirichlet Series and prime numbers is now evident (particularly by Euler's Product Identity), a relationship between prime numbers and $f(n)$ is only found when f is completely multiplicative. If this is the case, the Dirichlet Series for $F'(s)/F(s)$ from theorem 1.5.3 has the form:

$$\frac{F'(s)}{F(s)} = -\sum_{n=2}^{\infty} \frac{f(n)\Lambda(n)}{n^s}$$

(alternatively for the series $\log F(s)$ of theorem 1.5.5), where Λ is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with } p \text{ prime and } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

This takes a particularly simple form by taking ζ as F (that is, $f(n) = 1$). It is worthwhile to point out the fact that the preceding discussion is far from what historically happened. Actually, the inverse process took place, that is, the properties of the ζ function where discovered and they were later generalized to become the theory of Dirichlet Series introduced in the first chapter of this document. In any way however, one has

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}.$$

The partial sums of the coefficients of this series deserve their own name.

3.1.1 Definition. *Chebyshev's* ψ function is is defined by

$$\psi(x) = \sum_{n \le x} \Lambda(n).$$

The importance of $\psi(x)$ rests on the fact that if in fact $\pi(x) \sim x/\log x$, then $\psi(x)$ should behave like x. To see this (heuristically) note for a fixed prime $p \le x$, there are exactly $[\log_p x]$ powers of p less than x since $[\log_p x]$ is the greatest integer power of p less than x. Therefore, given that $\log_p x = \log x / \log p$,

$$\psi(x) = \sum_{n \le x} \Lambda(n) = \sum_{p \le x} \left[\frac{\log x}{\log p}\right] \log p.$$

In this way,

$$\psi(x) = \sum_{p \le x} \left[\frac{\log x}{\log p}\right] \log p \simeq \sum_{p \le x} \frac{\log x}{\log p} \log p = \log x \sum_{p \le x} 1 \simeq x$$

where the last approximation comes from the fact that if $\pi(x) \sim x/\log x$, then in a sum of the form $\sum_{p \le x}$ there are approximately $x/\log x$ terms. The following theorem is a rigorous proof if this fact.

3.1.2 Theorem. $\psi(x) \sim x$ *if and only if* $\pi(x) \sim \dfrac{x}{\log x}$.

Proof. By the previous arguments,

$$\psi(x) = \sum_{p \le x} \left[\frac{\log x}{\log p}\right] \log p \le \sum_{p \le x} \frac{\log x}{\log p} \log p = \pi(x) \log x. \tag{3.2}$$

Take $1 < y < x$ and notice that

$$\pi(x) = \pi(y) + \sum_{y < p \le x} 1 \le \pi(y) + \sum_{y < p \le x} \frac{\log p}{\log y} \le y + \frac{\psi(x)}{\log y}.$$

Therefore, putting $y = x/\log^2 x$ one obtains

$$\pi(x) \le \frac{x}{\log^2 x} + \frac{\psi(x)}{\log x - 2 \log\log x} \tag{3.3}$$

and combining the inequalities (3.2) and (3.3) it follows that

$$\frac{\psi(x)}{x} \le \frac{\pi(x)}{x/\log x} \le \frac{1}{\log x} + \frac{\psi(x)}{x} \frac{\log x}{\log x - 2\log\log x}$$

from which the statement follows. ∎

In what follows, an exposition showing that the properties of ζ'/ζ imply $\psi(x) \sim x$ will be given. The arguments are quite long and complicated. An essential part of the argument will require the use of the analyticity of $\zeta'(s)/\zeta(s)$ on the line Re $s = 1$, $s \neq 1$. This is the reason why the fact that $\zeta(1+it) \neq 0$ for $t \neq 0$ is of so much importance.

It is worthwhile mentioning the fact that shortly after the PNT was proved, it was shown that $\zeta(1+it) \neq 0$ for $t \neq 0$ is in fact equivalent to the PNT (see [Dia96]).

3.2 Technical Preliminaries

Some results from complex analysis will be needed for the proof of the Main Theorem. The proofs are given in this section. The following definition will be of great use to state the propositions more clearly.

3.2.1 Definition. Let $f(s)$ be a complex function, analytic in some open set containing the vertical line $L_c = \{s = c + it : t \in \mathbb{R}\}$. If

$$\int_c^{c+i\infty} f(s)ds = \int_0^\infty if(c+it)dt \quad \text{and} \quad \int_{c-i\infty}^{c} f(s)ds = \int_{-\infty}^{0} if(c+it)dt$$

converge, the integral of $f(s)$ over L_c is defined by

$$\int_{L_c} f(s)ds := \lim_{T\to\infty} \int_{c-iT}^{c+iT} f(s)ds = \lim_{T\to\infty} \int_{-T}^{T} if(c+it)dt$$

In what follows, the symbol $\int_{-\infty}^{\infty}$ will be used to mean $\lim_{T\to\infty} \int_{-T}^{T}$.

Integrals of the form previously defined are the ones which allow one to obtain information about the coefficients of a Dirichlet Series.

3.2.2 Lemma. *If C is simple closed positively oriented path containing $s = 0$ in its interior, then*

$$\int_C \frac{e^{as}}{s^2} ds = a2\pi i.$$

Proof. Since the Laurent series for e^{as}/s^2 centered at $s = 0$ is

$$\frac{e^{as}}{s^2} = \frac{1}{s^2} \sum_{n=0}^{\infty} \frac{(as)^n}{n!} = \frac{1}{s^2} + \frac{a}{s} + \sum_{n=2}^{\infty} \frac{a^n s^{n-2}}{n!},$$

the lema follows using theory of residues since the residue of e^{as}/s^2 at $s = 0$ is a. ■

3.2.3 Lemma. *If $c > 0$ then*

$$\frac{1}{2\pi i} \int_{L_c} \frac{x^s}{s^2} ds = \begin{cases} \log x & si\ x \geq 1 \\ 0 & si\ 0 < x < 1 \end{cases}$$

and if $c > 1$, then

$$\frac{1}{2\pi i} \int_{L_c} \frac{x^{s-1}}{(s-1)^2} ds = \begin{cases} \log x & si\ x \geq 1 \\ 0 & si\ 0 < x < 1. \end{cases}$$

Proof. The second statement follows from the first since by writing $c = 1 + b$ (with $b > 0$), one obtains

$$\frac{1}{2\pi i} \int_{L_c} \frac{x^{s-1}}{(s-1)^2} ds = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{ix^{b+it}}{(b+it)^2} dt = \frac{1}{2\pi i} \int_{L_b} \frac{x^s}{s^2} ds.$$

For the first statement, let C_R be the circle of radius $R > c$ with center $s = 0$. Let C_{1_R}, C_{2_R} be the segments of C_R which lie on the left and on the right of L_c respectively, and let L_R be the segment of L_c contained in the interior of C_R. Then $C_{1_R} \cup L_R$ and $C_{2_R} \cup L_R$ are simple closed paths and $\lim_{R\to\infty} L_R = L_c$.

For $x \geq 1$ consider the integral over $C_{1_R} \cup L_R$ with positive orientation. By lema 3.2.2 with $a = \log x$,

$$\frac{1}{2\pi i} \int_{C_{1_R} \cup L_R} \frac{x^s}{s^2} ds = \frac{1}{2\pi i} \int_{C_{1_R} \cup L_R} \frac{e^{s \log x}}{s^2} ds = \log x. \tag{3.4}$$

With the orientation of the path one also has $\lim_{R\to\infty} \int_{L_R} = \int_{L_c}$. Therefore, since by (3.4), $\log x = \int_{C_{1_R} \cup L_R} = \int_{C_{1_R}} + \int_{L_R}$, to complete the proof one only has to show that $\lim_{R\to\infty} \int_{C_{1_R}} = 0$. With this in mind, one finds a bound for the integrand on C_{1_R}.

If $s \in C_{1_R}$ then $\sigma \leq c$, and since $x \geq 1$,

$$\left| \frac{x^s}{s^2} \right| = \frac{x^\sigma}{|s^2|} = \frac{x^\sigma}{R^2} \leq \frac{x^c}{R^2}. \tag{3.5}$$

And so, since the length of C_{1_R} is less than $2\pi R$,

$$\left| \frac{1}{2\pi i} \int_{C_{1_R}} \frac{x^s}{s^2} ds \right| < \frac{1}{2\pi} \frac{x^c}{R^2} 2\pi R = \frac{x^c}{R}.$$

Therefore $\lim_{R\to\infty}\int_{C_{1_R}} = 0$ as desired.

In the case $0 < x < 1$ consider the integral over $C_{2_R} \cup L_R$ with positive orientation. It follows again that $\lim_{R\to\infty}\int_{L_R} = \int_{L_c}$. Since x^s/s^2 is analytic in $C_{2_R} \cup L_R$ and its interior,

$$\frac{1}{2\pi i}\int_{C_{2_R}\cup L_R} \frac{x^s}{s^2} ds = 0.$$

by Cauchy's theorem. It is therefore again sufficient to prove $\int_{C_{2_R}} \to 0$. However, since for $s \in C_{2_R}$ one has $\sigma \geq c$ and in this case $x < 1$, the bound (3.5) continues to hold. Therefore, $\lim_{R\to\infty}\int_{C_{2_R}} = 0$ which in turn implies $\int_{L_c} = 0$. ■

3.2.4 Lemma. *If* $c > 1$

$$\frac{1}{2\pi i}\int_{L_c} \frac{x^s}{s(s-1)} ds = \begin{cases} x-1 & \text{if } x \geq 1 \\ 0 & \text{if } 0 < x < 1 \end{cases}$$

and

$$\frac{1}{2\pi i}\int_{L_c} \frac{x^{s-1}}{s(s-1)} ds = \begin{cases} 1-1/x & \text{if } x \geq 1 \\ 0 & \text{if } 0 < x < 1. \end{cases}$$

Proof. The proof is similar to the previous one. First of all note that the second statement follows from the first since

$$\begin{aligned}\frac{1}{2\pi i}\int_{L_c} \frac{x^{s-1}}{s(s-1)} ds &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{ix^{c+it-1}}{(c+it)(c+it-1)} dt \\ &= \frac{1}{2\pi i}\frac{1}{x}\int_{-\infty}^{\infty} \frac{ix^c x^{it}}{(c+it)(c+it-1)} dt \\ &= \frac{1}{x}\frac{1}{2\pi i}\int_{L_c} \frac{x^s}{s(s-1)} ds.\end{aligned}$$

For the first statement, fix $c > 1$ and define C_R, C_{1_R}, C_{2_R} and L_R as in the previous lema. For all $s \in C_R$ one has $|s(s-1)| \geq R(R-1)$ by the inequality $|s_0 - s_1| \geq ||s_0| - |s_1||$.

For $x \geq 1$ consider $C_{1_R} \cup L_R$ with positive orientation and note that the integrand can be written as

$$\frac{x^s}{s(s-1)} = \frac{x^s}{s-1} - \frac{x^s}{s}.$$

Since $f(s) = x^s$ is analytic in, and in the interior of $C_{1_R} \cup L_R$, one obtains from Cauchy's theorem that

$$\frac{1}{2\pi i}\int_{C_{1_R}\cup L_R} \frac{f(s)}{s} ds = f(0) = 1 \quad \text{and} \quad \frac{1}{2\pi i}\int_{C_{1_R}\cup L_R} \frac{f(s)}{s-1} ds = f(1) = x.$$

Therefore,

$$\frac{1}{2\pi i}\int_{C_{1_R}\cup L_R} \frac{x^s}{s(s-1)} ds = x - 1.$$

Now, since for $s \in C_{1_R}$ $x \geq 1$ and $\sigma \leq c$,

$$\left|\frac{x^s}{s(s-1)}\right| = \frac{x^\sigma}{|s(s-1)|} \leq \frac{x^c}{R(R-1)}. \tag{3.6}$$

This implies,

$$\left|\frac{1}{2\pi i}\int_{C_{1_R}} \frac{x^s}{s(s-1)} ds\right| \leq \frac{1}{2\pi}\frac{x^c}{R(R-1)} 2\pi R = \frac{x^c}{R-1}.$$

And from this follows that

$$\frac{1}{2\pi i}\int_{L_c} \frac{x^s}{s(s-1)} = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{L_R} \frac{x^s}{s(s-1)} ds = x - 1.$$

in a manner similar to the proof of the previous lema.

In the case $0 < x < 1$, the integrand is analytic in $C_{2_R} \cup L_R$ which implies $\int_{C_{2_R}\cup L_R} = 0$ and (3.6) is also valid for $s \in C_{2_R}$. This allows to conclude that $\lim_{R\to\infty}\int_{C_{2_R}} = 0$, from which follows that $\int_{L_c} = 0$. ■

3.2.5 Lemma. *Let $g : \mathbb{R} \to \mathbb{C}$ with continuous derivative in $[1, x]$. Let $a : \mathbb{N} \to \mathbb{C}$ and define $A(x) = \sum_{n \le x} a(n)$. Then,*

$$\sum_{n \le x} a(n)[g(x) - g(n)] = \int_1^x A(t)g'(t)dt.$$

Proof. Using Abel's identity B.1 with $y = 1$,

$$\sum_{1 < n \le x} a(n)g(n) = A(x)g(x) - A(1)g(1) - \int_1^x A(t)g'(t)dt.$$

However, since $A(1) = a(1)$,

$$\int_1^x A(t)g'(t)dt = A(x)g(x) - \sum_{n \le x} a(n)g(n) = \sum_{n \le x} a(n)[g(x) - g(n)].$$

■

The following theorem establishes a relationship between the partial sums of the coefficients of a Dirichlet Series with the behaviour of Dirichlet series over vertical lines.

3.2.6 Theorem. *Let $F(s) = \sum f(n)/n^s$ be an absolutely convergent Dirichlet Series for $\sigma > 1$ and let*

$$S_f(x) = \sum_{n \le x} f(n).$$

One has the following equality when $c, x > 1$

$$\frac{1}{2\pi i} \int_{L_c} \frac{x^{s-1}}{s(s-1)} F(s)ds = \int_1^x \frac{S_f(y)}{y^2} dy.$$

Proof. Fix $x > 1$, $c > 1$ and define $G(s)$ and $H(s)$ by

$$G(s) = x^s \sum_{n \le x} \frac{f(n)}{n^s} = \sum_{n \le x} f(n) \left(\frac{x}{n}\right)^s$$

$$H(s) = x^s \sum_{n > x} \frac{f(n)}{n^s} = \sum_{n > x} f(n) \left(\frac{x}{n}\right)^s.$$

Note that $x^s F(s) = G(s) + H(s)$. The integrals for $G(s)$ and $H(s)$ will be evaluated separately. For the integral of $G(s)$ one obtains from lema 3.2.4 that

$$\frac{1}{2\pi i} \int_{L_c} \frac{G(s)}{s(s-1)} ds = \sum_{n \le x} \left(f(n) \frac{1}{2\pi i} \int_{L_c} \frac{(x/n)^s}{s(s-1)} ds \right) = \sum_{n \le x} \left[f(n) \left(\frac{x}{n} - 1 \right) \right]$$

since $x/n \ge 1$ for every term in the sum.

To calculate the integral of $H(s)$ consider the circle of radius $R > c$ with centre $s = 0$ and let C_R be the part of the circle on the right of L_c and let L_R the segment of L_c contained inside the circle. $H(s)$ is analytic in $L_R \cup C_R$ and its interior since $F(s)$ is analytic for $\sigma > 1$. Therefore, orienting $L_R \cup C_R$ positively one obtains

$$\int_{C_R \cup L_R} \frac{H(s)}{s(s-1)} ds = 0$$

and with this orientation one also has $\lim_{R \to \infty} \int_{L_R} = \int_{L_c}$. A proof that $\lim_{R \to \infty} \int_{C_R} = 0$ will be given. This will imply that $\int_{L_c} = 0$.

Let $A = x^c \sum_{n > x} |f(n)| / n^c$. Since $c > 1$, $A < \infty$ by the absolute convergence of $F(s)$. If $\sigma \ge c$,

$$|H(s)| = \left| \sum_{n > x} f(n) \left(\frac{x}{n}\right)^s \right| \le \sum_{n > x} |f(n)| \left|\frac{x}{n}\right|^\sigma \le \sum_{n > x} |f(n)| \left|\frac{x}{n}\right|^c = A$$

since in this case $x/n < 1$ for all terms of the series. Therefore, given that $\sigma \geq c$ for $s \in C_R$ one obtains

$$\left|\int_{C_R} \frac{H(s)}{s(s-1)} ds\right| \leq \frac{A}{R(R-1)} 2\pi R = \frac{2\pi A}{R-1}. \tag{3.7}$$

This implies that the integral in (3.7) tends to 0 when $R \to \infty$ and since $\int_{L_R \cup C_R} = \int_{L_R} + \int_{C_R}$,

$$\int_{L_c} \frac{H(s)}{s(s-1)} ds = \lim_{R\to\infty} \int_{L_R} \frac{H(s)}{s(s-1)} ds = 0.$$

Combining the results for $G(s)$ and $H(s)$ one concludes that

$$\frac{1}{2\pi i} \int_{L_c} \frac{x^s}{s(s-1)} F(s) ds = \sum_{n \leq x} \left[f(n) \left(\frac{x}{n} - 1 \right) \right]$$

and by an argument similar to the one used in the lema 3.2.4, this equation can be written as

$$\frac{1}{2\pi i} \int_{L_c} \frac{x^{s-1}}{s(s-1)} F(s) ds = \sum_{n \leq x} \left[f(n) \left(\frac{1}{n} - \frac{1}{x} \right) \right].$$

Note that the previous equality basically shows that the symbols of series and integral could be interchanged. To arrive at the desired equality one uses lema 3.2.5 with $g(y) = 1/y$ and $a(n) = f(n)$ to obtain

$$\sum_{n \leq x} \left[f(n) \left(\frac{1}{n} - \frac{1}{x} \right) \right] = -\sum_{n \leq x} f(n)[g(x) - g(n)] = \int_1^x \frac{S_f(y)}{y^2} dy.$$

■

3.2.7 Theorem (Riemann-Lebesgue Lema)**.** *Let $\phi : \mathbb{R} \to \mathbb{C}$ with continuous derivative and suppose that $\int_{-\infty}^{\infty} |\phi(t)|\, dt$ converges (in particular $\int_{-\infty}^{0} |\phi(t)|\, dt$ y $\int_{0}^{\infty} |\phi(t)|\, dt$ converge). Then, for $\lambda \in \mathbb{R}$*

$$\lim_{\lambda\to\infty} \int_{-\infty}^{\infty} e^{i\lambda t} \phi(t) dt = 0.$$

Proof. Let $\varepsilon > 0$. Since $\int_{-\infty}^{\infty} |\phi(t)|\, dt$ is convergent, there exists $T > 0$ such that $\int_{-\infty}^{-T} |\phi(t)|\, dt < \varepsilon/3$ and $\int_{T}^{\infty} |\phi(t)|\, dt < \varepsilon/3$. Therefore, for every λ one has

$$\left|\int_T^{\infty} e^{i\lambda t} \phi(t) dt\right| \leq \int_T^{\infty} \left|e^{i\lambda t} \phi(t)\right| dt = \int_T^{\infty} |\phi(t)|\, dt < \varepsilon/3$$
$$\left|\int_{-\infty}^{-T} e^{i\lambda t} \phi(t) dt\right| \leq \int_{-\infty}^{-T} |\phi(t)|\, dt < \varepsilon/3$$

Now, since $\phi'(t)$ is continuous in $[-T, T]$, the exists M such that $|\phi'(t)| \leq M$ for $t \in [-T, T]$. Taking

$$F_T(\lambda) := \int_{-T}^{T} e^{i\lambda t} \phi(t) dt,$$

and integrating by parts, one obtains

$$F_T(\lambda) = \frac{1}{i\lambda} \left(e^{i\lambda T} \phi(T) - e^{-i\lambda T} \phi(-T) - \int_{-T}^{T} e^{i\lambda t} \phi'(t) dt \right).$$

And so,

$$|F_T(\lambda)| \leq \frac{1}{\lambda} (|\phi(T)| + |\phi(-T)| + 2TM) = \frac{B}{\lambda}$$

with B independent of λ. Take λ_0 such that $|F_T(\lambda)| < \varepsilon/3$ for $\lambda > \lambda_0$. For $\lambda > \lambda_0$ one has

$$\left|\int_{-\infty}^{\infty} e^{i\lambda t} \phi(t) dt\right| < \varepsilon.$$

■

3.3 A General Theorem

All the tools for the proof of the Main Theorem have now been developed. The proof of the Main Theorem has been broken up into various steps to facilitate its reading. As will be seen, to prove the PNT one will have to bound in a very strict manner $\zeta'(\sigma+it)/\zeta(\sigma+it)$ in terms of t. First of all a lema.

3.3.1 Lemma. *Let $A:[1,\infty)\to[0,\infty)$ be a monotonic non decreasing function and suppose there exists an α such that*

$$\int_1^\infty \frac{A(x)-\alpha x}{x^2}dx$$

converges. Then, [1]

$$A(x)\sim\alpha x.$$

Proof. The proof will be given for $\alpha=1$ and then the general case will be proved.

Suppose there exists $\lambda>1$ such that $A(x)\geq\lambda x$ for some x. Given that $A(x)$ is increasing, one has

$$\int_x^{\lambda x}\frac{A(t)-t}{t^2}dt\geq\int_x^{\lambda x}\frac{\lambda x-t}{t^2}dt=\int_1^{\lambda}\frac{\lambda-t}{t^2}dt>0,$$

where the last equality is obtained with the substitution $t=xu$. Since the last integral is independent of x, the existence of infinite x_n with $x_n\to\infty$ for which $A(x_n)\geq\lambda x_n$ contradicts the convergence of the integral.

In a similar manner, the existence of $\lambda<1$ with $A(x)\leq\lambda x$ implies

$$\int_{\lambda x}^{x}\frac{A(t)-t}{t^2}dt\leq\int_{\lambda x}^{x}\frac{\lambda x-t}{t^2}dt=\int_{\lambda}^{1}\frac{\lambda-t}{t^2}dt<0$$

which again contradicts the convergence of the integral with the existence of a sequence x_n where $x_n\to\infty$ such that $A(x_n)\leq\lambda x_n$.

If $\alpha\neq 1$, notice that

$$\int_1^\infty\frac{A(x)-\alpha x}{x^2}dx=\alpha\int_1^\infty\frac{A(x)/\alpha-x}{x^2}dx.$$

Therefore, if the integral converges one would have $\lim_{x\to\infty}A(x)/\alpha x=1$, that is, $A(x)\sim\alpha x$. ∎

3.3.2 Theorem. *Let $F(s)$ be a complex function, analytic in an open region containing Re $s\geq 1$ except possibly by a simple pole at $s=1$ with residue α. If,*

1. *F admits a representation as a Dirichlet Series $\sum f(n)/n^s$ with f non negative in some open set containing Re $s\geq1$, and absolutely convergent for $\sigma>1$.*
2. *There exists $t_0\geq1$ and a function $P:[1,\infty)\to[0,\infty)$ such that*
   (a) $\displaystyle\int_1^\infty\frac{P(t)}{t^2}dt$ *converges.*
   (b) $|F(\sigma\pm it)|\leq P(t)$ *for $\sigma\geq1$, $t\geq t_0$*

Then,

$$\sum_{n\leq x}f(n)\sim\alpha x$$

Proof. By lema 3.3.1 it is enough to prove that

$$\int_1^\infty\frac{S_f(x)-\alpha x}{x^2}dx$$

where $S_f(x)=\sum_{n\leq x}f(n)$, converges. The following steps are a proof of this.
STEP 1: *Construction of ϕ.*

[1]Remember that $f(x)\sim g(x)$ means $\lim_{n\to\infty}f(x)/g(x)=1$.

Let $G(s) = F(s) - \alpha/(s-1)$ for $s \neq 1$. Defining G appropriately at $s = 1$ one makes $G(s)$ analytic in Re $s \geq 1$. Let b be the residue of the simple pole of $G(s)/(s-1)$ at $s = 1$ and define

$$h(s) = \frac{G(s)}{s-1} - \frac{b}{s-1}$$

for $s \neq 1$. Define $h(1)$ in such a way that $h(s)$ is analytic in Re $s \geq 1$. Let now $B = b - \alpha$ and define $\phi(s) = h(s)/s$. Therefore, $\phi(s)$ is analytic in Re $s \geq 1$ and one also has

$$\phi(s) = \frac{F(s)}{s(s-1)} - \frac{\alpha}{(s-1)^2} - \frac{B}{s(s-1)} \tag{3.8}$$

for $s \neq 1$ (this equality is not straight forward, but can be obtained after some algebra or partial fractions for example).

STEP 2: *Integral over L_c.*

For $x > 1$ and $c \geq 1$ define

$$I(x,c) = \frac{1}{2\pi i}\int_{L_c} x^{s-1}\phi(s)ds.$$

Fix $c > 1$. Using the expression for $x^{s-1}\phi(s)$ obtained from (3.8), that is,

$$x^{s-1}\phi(s) = \frac{x^{s-1}F(s)}{s(s-1)} - \frac{x^{s-1}\alpha}{(s-1)^2} - \frac{x^{s-1}B}{s(s-1)},$$

and the convergence of the integrals over L_c of each one of the terms, one obtains by the results in section 3.2 that for $c > 1$

$$\begin{aligned} I(x,c) &= \int_1^x \frac{S_f(y)}{y^2}dy - \alpha\log x - B\left(1 - \frac{1}{x}\right) \\ &= \int_1^x \frac{S_f(y) - \alpha y}{y^2}dy - B\left(1 - \frac{1}{x}\right). \end{aligned}$$

This shows that $I(x,c)$ is independent of c and that to obtain the result it is sufficient to prove that $I(x,c)$ converges when $x \to \infty$. To prove this, the line of integration will be shifted to $c = 1$ and the the Riemann-Lebesgue lema will be used. Since the expression for $I(x,c)$ has only been shown for $c > 1$, it will be necessary to prove that $I(x,c) = I(x,1)$.

STEP 3: *Bounds for $\phi(s)$.*

From (3.8) it follows that

$$|\phi(s)| \leq \frac{|F(s)| + |B|}{|s(s-1)|} + \frac{|\alpha|}{|s-1|^2}.$$

Taking $s = \sigma + it$ with $\sigma \geq 1$ and $|t| \geq t_0$ one obtains $|s(s-1)| \geq t^2$ and $|s-1|^2 \geq t^2$. therefore, defining $P_1(t) = P(t) + |B| + |\alpha|$, and using the hypothesis on $P(t)$ one obtains

$$|\phi(\sigma \pm it)| \leq \frac{P(t) + |B| + |\alpha|}{t^2} = \frac{P_1(t)}{t^2}.$$

Since $\int_1^\infty P_1(t)/t^2 dt$ converges for $\sigma \geq 1$, then $\int_{t_0}^\infty |\phi(\sigma + it)|\, dt$ and $\int_{-\infty}^{-t_0} |\phi(\sigma + it)|\, dt$ converge. In particular, given that $\phi(1+it)$ is continuous in $[-t_0, t_0]$,

$$\int_{-\infty}^{\infty} |\phi(1+it)|\, dt < \infty.$$

STEP 4: $I(x,c) = I(x,1)$ *for* $c > 1$.

Let $g(s) = x^{s-1}\phi(s)$. To prove that $I(x,c) = I(x,1)$ consider the rectangular path R with vertices $c \pm it$ and $1 \pm it$. Give R positive orientation and let V_c, V_1 be the vertical segments of R at $\sigma = c$ and $\sigma = 1$ respectively. Similarly let H_{-t}, H_t be the horizontal segments. Notice that $\lim_{t\to\infty}\int_{V_c} g(s)ds = I(x,c)$ and $\lim_{t\to\infty} -\int_{V_1} g(s)ds = I(x,1)$.

Since $g(s)$ is analytic in R and its interior,

$$\int_R g(s)ds = 0.$$

Therefore,

$$-\int_{V_1} g(s)ds = \int_{V_c} g(s)ds + \int_{H_{-t}} g(s)ds + \int_{H_t} g(s)ds.$$

So, to prove the equality it is sufficient to prove that $\int_{H_t} g(s)ds$ and $\int_{H_{-t}} g(s)ds$ tend to zero when $t \to \infty$. The proof of this fact will be given for $\int_{H_t}$. The other case is similar.

Notice that

$$\int_{H_t} g(s)ds = \int_{c+it}^{1+it} g(s)ds = -\int_1^c ig(\sigma + it)d\sigma = -\int_1^c ix^{\sigma+it-1}\phi(\sigma + it)d\sigma.$$

Therefore, for $\sigma \geq 1$ and $t \geq t_0$ taking $A = \int_1^c x^{\sigma-1}d\sigma$ one has

$$\left|\int_{H_t} g(s)ds\right| \leq \int_1^c x^{\sigma-1}\left|\phi(\sigma + it)\right| d\sigma \leq A\frac{P_1(t)}{t^2}.$$

Since $\int_1^\infty P_1(t)/t^2 dt$ converges, $\lim_{t\to\infty} P_1(t)/t^2 = 0$ and the result follows. That is, $I(x,c) = I(x,1)$.

STEP 5: *Application of Riemann-Lebesgue.*

Notice that

$$I(x,c) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} ix^{c-1+it}\phi(c+it)dt = \frac{x^{c-1}}{2\pi i}\int_{-\infty}^{\infty} ie^{it\log x}\phi(c+it)dt.$$

Therefore,

$$I(x,1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\log x}\phi(1+it)dt.$$

Since $\int_{-\infty}^{\infty} \left|\phi(1+it)\right| dt < \infty$, the Riemann-Lebesgue lema implies that $\lim_{x\to\infty} I(x,c) = \lim_{x\to\infty} I(x,1) = 0$ and the theorem is proved. ■

3.4 Proof of the PNT

The theorem from the previous section will be used to prove the PNT. For this, it will be sufficient to prove $\psi(x) \sim x$. This asymptotic identity is a direct consequence in the Main Theorem from the previous section if it is proved that the hypothesis from 3.3.2 are satisfied and $\alpha = 1$ for the function $-\zeta'(s)/\zeta(s)$.

First of all a lema from complex analysis to prove that $\alpha = 1$.

3.4.1 Lemma. *If f has a pole of order k at $s = \alpha$ then $f'(s)/f(s)$ has a simple pole (order 1) with residue $-k$ at $s = \alpha$.*

Proof. Write $f(s) = g(s)/(s-\alpha)^k$ in some punctured disc around α with g analytic at α and $g(\alpha) \neq 0$. Differentiating,

$$f'(s) = \frac{g'(s)}{(s-\alpha)^k} - \frac{kg(s)}{(s-\alpha)^{k+1}} = f(s)\left(\frac{-k}{s-\alpha} + \frac{g'(s)}{g(s)}\right).$$

Therefore,

$$\frac{f'(s)}{f(s)} = \frac{-k}{s-\alpha} + \frac{g'(s)}{g(s)}.$$

Since $g(\alpha) \neq 0$, the function $g'(s)/g(s)$ is analytic at α and the lema follows. ■

3.4.2 Corollary. *$-\zeta'(s)/\zeta(s)$ is analytic in Re $s \geq 1$ except at $s = 1$ where it has a simple pole with residue 1.*

Proof. Analyticity for Re $s > 1$ was proved in 2.1.7. Analyticity at Re $s = 1$ with $s \neq 1$ follows from the fact that $\zeta(1+it) \neq 0$. The pole and its residue follow from the lama since ζ has a simple pole at $s = 1$ with residue 1. ∎

The following result will be proved in the next section. Assuming it, the proof of the PNT is immediate. The details are given in the next section so that the final step in the proof of the PNT is not obscured, since the details of the proof of the next result are even more technical in nature that what has already been done.

3.4.3 Theorem. *There exists a constant K such that for $\sigma \geq 1$ and $t \geq e^5$,*

$$\left|\frac{\zeta'(\sigma \pm it)}{\zeta(\sigma \pm it)}\right| \leq K \log^9 t.$$

3.4.4 Theorem.

$$\int_1^\infty \frac{\log^9 t}{t^2}\, dt \quad \textit{converges}.$$

Proof. With integration by parts with $u = \log^9 t$ and $dv = 1/t^2$. ∎

3.4.5 Theorem (The Prime Number Theorem)**.**

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

Proof. Take the function $F(s) = -\zeta'(s)/\zeta(s)$ that satisfies the hypothesis of theorem 3.3.2 by 3.4.2, 3.4.4 and 2.1.7. Therefore,

$$\psi(x) \sim x$$

that by 3.1.2 is equivalent to the PNT. ∎

3.5 Bounds on $\zeta(\sigma \pm it)$

The purpose of this section will be to prove that $|\zeta'(\sigma \pm it)/\zeta(\sigma \pm it)| \leq K \log^9 t$ in a certain region. This will fill in the missing details in the proof of the PNT. For obtaining this bound, the integral representations of $\zeta(s)$ and $\zeta'(s)$ from theorems 2.3.6 and 2.3.7 will be indispensable.

The following lema shows that all bounds on $\zeta(\sigma + it)$ will also be valid for $\zeta(\sigma - it)$ (and also for ζ').

3.5.1 Lemma. $\overline{\zeta(s)} = \zeta(\overline{s})$.

Proof.

$$n^{-\overline{s}} = e^{-\overline{s}\log n} = e^{-\sigma\log n} e^{it\log n} = \overline{e^{-\sigma \log n} e^{-it\log n}} = \overline{n^{-s}}.$$

∎

3.5.2 Lemma. *If $N \geq 4$, then*

$$\sum_{n=1}^{N} \frac{1}{n} \leq 2\log N \qquad \textit{and} \qquad \sum_{n=1}^{N} \frac{\log n}{n} \leq 2\log^2 N.$$

Proof. By theorem B.3 with $f(x) = 1/x$,

$$\sum_{n=1}^{N} \frac{1}{n} \leq 1 + \int_1^N \frac{1}{x} dx = 1 + \log N \leq 2\log N.$$

For the other series use the theorem with $f(x) = (\log x)/x$. However, in this case $f(x)$ is decreasing only if $x > e$. Therefore,

$$\sum_{n=4}^{N} \frac{\log n}{n} \leq \int_3^N \frac{\log x}{x} dx = \frac{1}{2}(\log^2 N - \log^2 3).$$

This implies,

$$\sum_{n=1}^{N} \frac{\log n}{n} \leq \log^2 N + (\log 2)/2 + (\log 3)/3 \leq 2\log^2 N.$$

∎

3.5.3 Theorem. *If* $\sigma \geq 1$ *y* $t \geq e^3$ *then* $|\zeta(\sigma + it)| \leq 3 \log t$.

As will be clear in the proof, this is not the best bound possible with the given arguments. What is important for the main purpose is just to know that one can bound $\zeta(\sigma + it)$ in terms of t from some value of t onwards.

Proof. Fix $s = \sigma + it$ with σ and t satisfying the hypothesis. Let $N = [t]$ and consider the equality

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s \int_N^\infty \frac{x - [x]}{x^{s+1}} dx. \tag{3.9}$$

from theorem 2.3.6. Under the hypothesis on σ and t one obtains the following bounds for (3.9) using lema 3.5.2 and the fact that $N \leq t$:

$$\begin{aligned} \left| \sum_{n=1}^{N} \frac{1}{n^s} \right| &\leq \sum_{n=1}^{N} \frac{1}{n^\sigma} \leq \sum_{n=1}^{N} \frac{1}{n} \leq 2 \log N \leq 2 \log t \\ \left| \frac{N^{1-s}}{s-1} \right| &\leq \frac{1}{|s-1|} \leq \frac{1}{t} \\ \left| s \int_N^\infty \frac{x-[x]}{x^{s+1}} dx \right| &\leq |s| \int_N^\infty \frac{1}{x^{\sigma+1}} dx = \frac{|s|}{\sigma N^\sigma} \leq \frac{\sigma + t}{\sigma N} \leq 2. \end{aligned}$$

Therefore,

$$|\zeta(\sigma + it)| \leq 2 \log t + 1/t + 2 \leq 3 \log t.$$

■

3.5.4 Theorem. *If* $\sigma \geq 1$ *and* $t \geq e^5$ *then* $|\zeta'(\sigma + it)| \leq 4 \log^2 t$.

Proof. The proof is similar. Fix $s = \sigma + it$ with σ and t satisfying the hypothesis, take $N = [t]$ and use the expression for $\zeta'(\sigma + it)$ from theorem 2.3.7

$$\begin{aligned} \zeta'(s) = &- \sum_{n=1}^{N} \frac{\log n}{n^s} + s \int_N^\infty \frac{(x - [x]) \log x}{x^{s+1}} dx - \int_N^\infty \frac{x - [x]}{x^{s+1}} dx \\ &- \frac{N^{1-s} \log N}{s-1} - \frac{N^{1-s}}{(s-1)^2} \end{aligned}$$

The following bound for each term are obtained:

$$\begin{aligned} \left| \sum_{n=1}^{N} \frac{\log n}{n^s} \right| &\leq \log t \sum_{n=1}^{N} \frac{1}{n} \leq 2 \log^2 t \\ \left| \frac{N^{1-s} \log N}{s-1} \right| &\leq \frac{N^{1-\sigma} \log N}{|s-1|} \leq \frac{\log t}{t} < 2 \\ \left| \frac{N^{1-s}}{(s-1)^2} \right| &\leq \frac{1}{t^2} \leq \frac{1}{2} \\ \left| s \int_N^\infty \frac{(x-[x]) \log x}{x^{s+1}} dx \right| &\leq |s| \left(\frac{\log N}{\sigma N^\sigma} + \frac{1}{\sigma^2 N^\sigma} \right) \leq (\log N + 1) \frac{\sigma + t}{\sigma N} \\ &\leq 2 \log t + 2 \\ \left| \int_N^\infty \frac{x - [x]}{x^{s+1}} dx \right| &\leq \int_N^\infty \frac{1}{x^2} dx = \frac{1}{N} \leq \frac{1}{2}. \end{aligned}$$

Therefore,

$$|\zeta'(s)| \leq 2 \log^2 t + 5 + 2 \log t \leq 4 \log^2 t.$$

■

3.5.5 Theorem. *For* $\sigma \geq 1$ *y* $t \geq e^5$ *there exists a constant* K *such that*

$$\frac{1}{|\zeta(\sigma + it)|} \leq K \log^7 t.$$

It is known that $\zeta(1+it)$ comes arbitrarily close to zero, therefore, the bound cannot be replaced by a constant. One should compare this with the result from theorem 2.4.3.

Proof. Suppose that $\sigma \leq 2$. This will cause no harm to the proof since if $\sigma > 2$ then,

$$\left|\frac{1}{\zeta(s)}\right| = \left|\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2)$$

(the value of $\zeta(2)$ is $\pi^2/6$, as first shown by Euler), so K can be found such that the bound is also valid in this case (once K is found for $1 \leq \sigma \leq 2$ one takes the largest of the two).

Suppose then that $1 \leq \sigma \leq 2$, $t \geq e^5$ and define $g_1(t) = 3\log t$, $g_2(t) = 4\log^2 t$. Using the expression for ζ from theorem 2.3.3 and ignoring the value of the integral one obtains that for $1 < \sigma \leq 2$

$$|\zeta(\sigma)| \leq 1 + \frac{1}{\sigma - 1} \leq \frac{2}{\sigma - 1}. \tag{3.10}$$

From now on it will be assumed that $\sigma \neq 1$ so that the inequality (3.10) is satisfied. The result for $\sigma = 1$ will follow by continuity. By (3.10), 2.4.2 and 3.5.3 one has,

$$1 \leq \zeta(\sigma)^3 \left|\zeta(\sigma+it)\right|^4 \left|\zeta(\sigma+i2t)\right| \leq \frac{2^3}{(\sigma-1)^3} g_1(2t) \left|\zeta(\sigma+it)\right|^4.$$

Therefore, defining

$$f(\sigma, t) := \frac{(\sigma-1)^{3/4}}{2^{3/4}[g_1(2t)]^{1/4}}$$

it follows that for $\sigma \in (1, 2]$ and $t \geq e^5$,

$$|\zeta(\sigma+it)| \geq f(\sigma, t).$$

Take $\alpha \in (1, 2)$. If $1 < \sigma \leq \alpha \leq 2$ notice that

$$|\zeta(\sigma+it) - \zeta(\alpha+it)| \leq \int_\sigma^\alpha |\zeta'(x+it)|\, dx \leq \int_1^\alpha |\zeta'(x+it)|\, dx \leq g_2(t)(\alpha-1).$$

Therefore, by the triangle inequality,

$$\begin{aligned}
|\zeta(\sigma+it)| &\geq |\zeta(\alpha+it)| - |\zeta(\sigma+it) - \zeta(\alpha+it)| \\
&\geq |\zeta(\alpha+it)| - (\alpha-1)g_2(t) \\
&\geq f(\alpha, t) - (\alpha-1)g_2(t).
\end{aligned}$$

If $\alpha \leq \sigma$ the previous inequality if also valid since $|\zeta(\sigma+it)| \geq f(\sigma,t) \geq f(\alpha,t)$, $(\sigma-1)^{3/4} \geq (\alpha-1)^{3/4}$ and $g_2(t) > 0$.

The previous arguments show that for $\sigma \in (1, 2]$, $t \geq e^5$ and $\alpha \in (1, 2)$ it is true that,

$$|\zeta(\sigma+it)| \geq f(\alpha, t) - (\alpha-1)g_2(t). \tag{3.11}$$

Make now α depend on t in such a way that $f(\alpha,t) - (\alpha-1)g_2(t) = (\alpha-1)g_2(t)$. This is accomplished by taking

$$\alpha - 1 = \frac{1/2^7}{g_1(2t)[g_2(t)]^4}.$$

This implies that $\alpha > 1$ and since $g_1(2t), g_2(t) \geq 1$ then $\alpha < 2$. Therefore, inequality (3.11) is satisfied with this α and so,

$$|\zeta(\sigma+it)| \geq (\alpha-1)g_2(t) = \frac{C}{g_1(2t)[g_2(t)]^3} \geq \frac{D}{\log^7 t}$$

for some D as desired. ■

By theorems 3.5.4, 3.5.5 and lema 3.5.1, there exists a constant K such that for $\sigma \geq 1$ and $t \geq e^5$,

$$\left|\frac{\zeta'(\sigma \pm it)}{\zeta(\sigma \pm it)}\right| \leq K \log^9 t$$

as was to be proved.

3.6 Some comments about the PNT

The PNT was first conjectured (publicly) by Legendre int the year 1798 in a form different from 3.4.5 and somewhat incorrect. Gauss however, states in a letter to Encke in the year 1849 that he had already conjectured the PNT in 1792 (a translation from this letter may be found at [Gol73]). Gauss states in the letter that[2]

$$Li(x) := \int_2^x \frac{1}{\log t} dt$$

was a good approximation of $\pi(x)$. The following theorem shows that Gauss's conjecture was correct.

3.6.1 Theorem. *$Li(x) \sim \pi(x)$*

Proof. Since $\lim_{x\to\infty} Li(x) = \infty$,

$$\lim_{x\to\infty} \frac{Li(x)}{x/\log x} = \lim_{x\to\infty} \frac{\log x}{\log x - 1} = 1$$

by l'Hospital's rule. It is easy to see that the relation $f \sim g$ is transitive, so the result follows from the form of the PNT in 3.4.5. ∎

Given that the approximations obtained by the number theorem concern quotients of quantities, it is natural to wonder what is the real error in approximating $\pi(x)$ by $x/\log x$ or by $Li(x)$, after all, one has $x^2 + x^\delta \sim x^2$ for all $\delta < 2$ and so the error can be considerable. It has been shown that $Li(x)$ is a better approximation of $\pi(x)$ than $x/\log x$ is. There is also an intriguing result which relates the order of the error to the Riemann Hypothesis. More precisely, the following statement is equivalent to Riemann's Hypothesis: For every $\epsilon > 0$ there exists a constant K such that

$$|Li(x) - \pi(x)| \leq K x^{1/2+\epsilon}.$$

Therefore, the size of the error of our approximation has a very close relation to Riemann's Hypothesis. The following equality, known *Riemann's explicit formula* as also exhibits this relationship :

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - 1/x^2)$$

(here, of course, ζ has been continued to an open set which contains $s = 0$) where the series is taken over all the zeros ρ of ζ with positive real part and $\psi(x)$ is *Chebyshev's* function as defined in 3.1.1.

The following are a few consequences that can be deduced quite easily from the PNT.

3.6.2 Theorem. *If p_n is the n-th prime, then $p_n \sim n \log n$.*

Proof. Since $\pi(p_n) = n$, the PNT implies that

$$\lim_{n\to\infty} \frac{n \log p_n}{p_n} = 1. \tag{3.12}$$

In this manner,

$$\lim_{n\to\infty} (\log n + \log\log p_n - \log p_n) = 0$$

which dividing by $\log p_n$ becomes

$$\lim_{n\to\infty} \frac{\log n + \log\log p_n}{\log p_n} = 1.$$

However, using l'Hospital's rule for an appropriate differentiable function P with $P(n) = p_n$ one obtains $\lim_{n\to\infty} \log\log p_n/\log p_n = 0$, so

$$\lim_{n\to\infty} \frac{\log n}{\log p_n} = 1. \tag{3.13}$$

The theorem follows by multiplying (3.12) and (3.13). ∎

[2]Some authors define $Li(x)$ as

$$Li(x) = \lim_{\epsilon\to 0^+} \left\{ \int_0^{1-\epsilon} + \int_{1+\epsilon}^x \right\} \frac{1}{\log t} dt.$$

It can be shown that the different definitions differ in no more than 1.1 and so are equivalent in this context.

3.6.3 Corollary.

$$\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1.$$

3.6.4 Corollary. *The set $S := \{p/q : p \text{ and } q \text{ are primes}\}$ is dense $\mathbb{R}^+$.*

Proof. It is enough to prove that S is dense in $\mathbb{Q}^+ = \mathbb{Q} \cap (0, \infty)$. Let $a/b \in \mathbb{Q}^+$. Then, taking again p_n as the n-th prime, it follows by 3.6.2 that

$$\lim_{n\to\infty} \frac{p_{an}}{p_{bn}} = \lim_{n\to\infty} \frac{a\log(an)}{b\log(bn)} = \frac{a}{b}.$$

■

Appendix A

Complex Integration

A.1 Theorem. *Let $V \subseteq \mathbb{C}$ be an open set and $f_n : V \to \mathbb{C}$ a sequence of analytic functions on V which converge point-wise to $f : V \to \mathbb{C}$. If $f_n \to f$ uniformly on every compact subset of V, then f is analytic and $f_n' \to f'$ in V.*

Proof. By Morera's theorem (see [Bro92], pg. 141), if U is open and

$$\int_C f(z)dz = 0$$

for every simple closed path contained in U, then f is analytic in U. Take $z_0 \in V$ and let $B(z_0) \subset V$ be an open ball around z_0 with $\overline{B(z_0)} \subseteq V$. By hypothesis, $f_n \to f$ uniformly in $\overline{B(z_0)}$, so if $\epsilon > 0$ there exists N such that $|f(z) - f_n(z)| < \epsilon$ for every $n \geq N$ and $z \in \overline{B(z_0)}$. Therefore, for every simple close path C contained in $B(z_0)$ and every $n \geq N$,

$$\left| \int_C f(z)dz - \int_C f_n(z)dz \right| = \left| \int_C f(z) - f_n(z)dz \right| \leq \epsilon L$$

where L is the length of C. This implies that

$$\int_C f(z)dz = \lim_{n\to\infty} \int_C f_n(z)dz = 0$$

since $\int_C f_n(z)dz = 0$ by Cauchy's theorem ([Bro92], pgs. 136-140) and so f is analytic in $B(z_0)$, and so in V.

By Cauchy's integral formula,

$$f'(z_0) = \frac{1}{2\pi i} \int_{\partial B(z_0)} \frac{f(z)}{(z - z_0)^2} dz \qquad \textit{and} \qquad f_n'(z_0) = \frac{1}{2\pi i} \int_{\partial B(z_0)} \frac{f_n(z)}{(z - z_0)^2} dz$$

where $\partial B(z_0)$ is the boundary of $B(z_0)$ with positive orientation. Therefore,

$$|f'(z_0) - f_n'(z_0)| \leq \left| \frac{1}{2\pi i} \int_C \frac{f(z) - f_n(z)}{(z - z_0)^2} dz \right| \leq \frac{\epsilon}{R}$$

for every $n \geq N$ where R is the radius of $\partial B(z_0)$ since

$$\left| \frac{f(z) - f_n(z)}{(z - z_0)^2} \right| \leq \frac{\epsilon}{R^2}.$$

From this it follows that $f_n'(z_0) \to f'(z_0)$. ■

Appendix B

Relationships between Integrals and Series

For analytic number theory, it is essential to know that relationships between series and integrals. The following theorem will be of great use.

B.1 Theorem (Abel's Identity)**.** *Consider* $a : \mathbb{N} \to \mathbb{C}$ *and define* $A(x) = \sum_{n \leq x} a(n)$ *if* $x \geq 1$, 0 *otherwise. Let* $0 < y < x$ *and* $f : \mathbb{R} \to \mathbb{C}$ *be a function with continuous derivative in* $[y, x]$*.* Abel's Identity *states that:*

$$\sum_{y<n\leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt.$$

Notice the similarity with the formula for integration by parts.

Proof. Write $[x]$ for the greatest integer less than or equal to x. Now,

$$\begin{aligned}
\sum_{y<n\leq x} a(n)f(n) &= \sum_{n=[y]+1}^{[x]} a(n)f(n) \\
&= \sum_{n=[y]+1}^{[x]} (A(n) - A(n-1))f(n) \\
&= \sum_{n=[y]+1}^{[x]} A(n)f(n) - \sum_{n=[y]+1}^{[x]} A(n-1)f(n) \\
&= \sum_{n=[y]+1}^{[x]} A(n)f(n) - \sum_{n=[y]}^{[x]-1} A(n)f(n+1) \\
&= A([x])f([x]) - A([y])f([y+1]) - \sum_{n=[y]+1}^{[x]-1} A(n)(f(n+1) - f(n)) \\
&= A([x])f([x]) - A([y])f([y+1]) - \sum_{n=[y]+1}^{[x]-1} \int_n^{n+1} A(n)f'(t)dt \\
&= A([x])f([x]) - A([y])f([y+1]) - \int_{[y]+1}^{[x]} A(t)f'(t)dt.
\end{aligned}$$

Substituting $A([x])f([x])$ and $A([y])f([y]+1)$ in the last line by the expressions which follow from (1) and (2), (shown below) one obtains the desired equality. Note that $A([x]) = A(x)$.

$$\int_{[x]}^x A(t)f'(t)dt = A([x]) \int_{[x]}^x f'(t)dt = A(x)f(x) - A([x])f([x]) \tag{1}$$

$$\int_y^{[y]+1} A(t)f'(t)dt = A([y]) \int_y^{[y]+1} f'(t)dt = A([y])f([y]+1) - A(y)f(y) \tag{2}$$

■

B.2 Theorem (Euler's Summation Formula)**.** *Let* $f:\mathbb{R}\to\mathbb{C}$ *have continuous derivative in* $[y,x]$*,* $0<y<x$*. Then,*

$$\sum_{y<n\le x} f(n) = \int_y^x f(t)dt + \int_y^x (t-[t])f'(t)dt + f(x)([x]-x) - f(y)([y]-y).$$

Proof. Applying Abel's identity B.1 with $a(n)=1$ which implies $A(x)=[x]$, one obtains

$$\sum_{y<n\le x} f(n) = f(x)[x] - f(y)[y] - \int_y^x [t]f'(t)dt \tag{3}$$

however, integrating by parts,

$$\int_y^x f(t)dt = xf(x) - yf(y) - \int_y^x tf'(t)dt \tag{4}$$

Subtracting (4) from (3) and rearranging terms one obtains the desired equality. ■

B.3 Theorem. *Let* $n_1<n_2$ *be integers. If* $f:[n_1,n_2]\to[0,\infty)$ *is continuous and monotonically decreasing then*

$$\sum_{n=n_1+1}^{n_2} f(n) \le \int_{n_1}^{n_2} f(t)dt \le \sum_{n=n_1}^{n_2-1} f(n).$$

Proof. Take an integer r with $n_1<r\le n_2$ and let $t\in\mathbb{R}$ be such that $r-1\le t\le r$. Since f is decreasing, $f(r-1)\ge f(t)\ge f(r)$. Therefore, integrating from $r-1$ to r with respect to t one obtains

$$f(r) \le \int_{r-1}^{r} f(t)dt \le f(r-1).$$

The inequality is obtained by summing for $r=n_1+1, n_1+2, \ldots, n_2$. ■

B.4 Corollary. *Let* $f:[1,\infty)\to[0,\infty)$ *be a continuous decreasing function. Then* $\sum_{n=1}^{\infty} f(n)$ *converges if and only if* $\int_1^\infty f(t)dt$ *converges.*

Appendix C

Infinite Products

The contents of this appendix are taken from [Jam03] pgs. 228-229.

C.1 Definition. Let $\{a_n\}_{n\in\mathbb{N}}$ be a complex sequence. The infinite product

$$\prod_{n=1}^{\infty}(1+a_n) \tag{1}$$

is said to converge to P if the sequence $\{p_n\}_{n\in\mathbb{N}}$ defined by

$$p_n = \prod_{i=1}^{n}(1+a_i) \tag{2}$$

converges to P. In such a case, the value of the infinite product is said to be P.

The symbol $\prod(1+a_n)$ will be used to refer to (1) whenever no ambiguities can arise.

C.2 Theorem. *If $\prod(1+a_n)$ and $\prod(1+b_n)$ converge, then*

$$\prod_{n=1}^{\infty}[(1+a_n)(1+b_n)] = \prod_{n=1}^{\infty}(1+a_n)\prod_{n=1}^{\infty}(1+b_n).$$

Moreover, if $\prod(1+a_n) \neq 0$ then,

$$\prod_{n=1}^{\infty}\frac{1}{(1+a_n)} = \frac{1}{\prod_{n=1}^{\infty}(1+a_n)}$$

Proof. Direct consequence of the definition. Note that if $\prod(1+a_n)\neq 0$ then $a_n \neq 1$ for all n. ■

C.3 Theorem. *If $\sum a_n$ converges absolutely, then $\prod(1+a_n)$ converges.*

Proof. For every $x \geq 0$ one has

$$1+x \leq e^x$$

by the Taylor series of e^x. Therefore, for every $i \in \mathbb{N}$

$$|1+a_i| \leq 1+|a_i| \leq e^{|a_i|}. \tag{3}$$

Define p_n as in (2) and let $S = \sum |a_n|$. By (3)

$$|p_n| \leq e^{\sum_{i=1}^{n}|a_i|} \leq e^S.$$

So $|a_n p_n| \leq e^s |a_n|$ and

$$\sum_{n=2}^{\infty}|a_n p_{n-1}| \leq Se^s.$$

In particular, $\sum_{n=2}^{\infty} a_n p_{n-1}$ converges. However, $a_n p_{n-1} = p_n - p_{n-1}$ so $\sum_{n=2}^{\infty}(p_n - p_{n-1})$ converges. Since $\sum_{n=2}^{\infty}(p_n - p_{n-1}) = \lim_{n\to\infty}(p_n - p_1)$ it follows that $\lim_{n\to\infty} p_n$ exists and therefore $\prod(1+a_n)$ converges. ■

C.4 Theorem. *If $\sum a_n$ converges absolutely and $a_n \neq -1$ for every n, then $\prod(1 + a_n) \neq 0$.*

Proof. Let $P = \prod(1+a_n)$. The convergence of the infinite product $\prod\left(\frac{1}{1+a_n}\right)$ will be proved. This will imply the theorem by C.2.

Take

$$b_n = \frac{a_n}{1 + a_n}.$$

Since $\lim_{n\to\infty} a_n = 0$ by the absolute convergence of $\sum a_n$, there exists $N \in \mathbb{N}$ such that $|1 + a_n| \geq 1/2$ for $n \geq N$. Therefore, $|b_n| \leq 2\,|a_n|$ for $n \geq N$ which implies that $\sum b_n$ converges absolutely. By the previous theorem it follows that $\prod(1 - b_n)$ converges. However,

$$1 - b_n = \frac{1}{1 + a_n}$$

and so, $\displaystyle\prod_{n=1}^{\infty} \frac{1}{1 + a_n}$ converges and the theorem is proved. ■

Appendix D

Functions defined by Dirichlet Integrals

The content of this appendix is taken from [Jam03] pgs. 230-231.

D.1 Lemma. *If* $|s| \leq 1/2$*, then* $|e^s - 1 - s| \leq |s|^2$

Proof. From the Taylor series of e^s with centre at $s = 0$ one has:

$$|e^s - 1 - s| \leq \sum_{n=2}^{\infty} \frac{|s^n|}{n!} \leq \frac{|s|^2}{2} \sum_{n=0}^{\infty} |s|^n = \frac{|s|^2}{2(1-|s|)} \leq |s|^2 .$$

■

D.2 Theorem. *Let* $0 < a < b < \infty$*,* $f : [a,b] \to \mathbb{C}$ *bounded and Riemann Integrable in* $[a,b]$ *and define* $I : \mathbb{C} \to \mathbb{C}$ *by*

$$I(s) = \int_a^b \frac{f(x)}{x^s} dx.$$

Then, $I(s)$ *is analytic in all of* $\mathbb{C}$ *and*

$$I'(s) = -\int_a^b \frac{f(s) \log x}{x^s} dx$$

Note that the integrand in $I'(s)$ is simply the derivative of the integran in $I(s)$. The theorem states that one can interchange the symbols of differentiation and integration for these functions.

Proof. Take $s_0 \in \mathbb{C}$ and define $I'(s)$ as before (It of course still has no relation with the derivative of $I(s)$). What has to be proved is that

$$\lim_{s \to s_0} \frac{I(s) - I(s_0)}{s - s_0} = I'(s_0).$$

By the definitions of $I(s)$ and $I'(s)$ one has that

$$I(s) - I(s_0) - (s - s_0)I'(s_0) = \int_a^b \left(\frac{f(x)}{x^s} - \frac{f(x)}{x^{s_0}} + (s - s_0)\frac{f(x)\log x}{x^{s_0}} \right) dx.$$

Denote by $g(s, s_0)$ the integrand of the previous equation. Then

$$|g(s, s_0)| = \frac{|f(x)|}{|x^{s_0}|} \left| \frac{1}{x^{s-s_0}} - 1 + (s - s_0)\log x \right| .$$

Let M be such that $|f(x)| \leq M$ and K such that $|\log x| \leq K$ in $[a, b]$. By lema D.1, for $s \in \mathbb{C}$ with $K|s - s_0| \leq 1/2$,

$$\begin{aligned} \left| \frac{1}{x^{s-s_0}} - 1 + (s - s_0)\log x \right| &= \left| e^{(s_0 - s)\log x} - 1 - (s_0 - s)\log x \right| \\ &\leq |s - s_0|^2 \log^2 x. \end{aligned}$$

Therefore,

$$|g(s,s_0)| \le \frac{M}{x^{\sigma_0}} |s-s_0|^2 \log^2 x.$$

And so, defining $A = M \int_a^b \frac{\log^2 x}{x^{\sigma_0}} dx$ one has,

$$|I(s) - I(s_0) - (s-s_0)I'(s_0)| \le A |s-s_0|^2 .$$

That is,

$$\left| \frac{I(s) - I(s_0)}{s - s_0} - I'(s_0) \right| \le A |s - s_0| .$$

Taking $s \to s_0$ the result follows since A is independent from s. ∎

D.3 Theorem. *Let $N \in \mathbb{N}$, $N \neq 0$ and $f : [N, \infty) \to \mathbb{C}$ be a continuous function except possibly at $\mathbb{N} \cap [N, \infty)$ where, however, left and right limits exist. Suppose also that there exist $M, \alpha \in \mathbb{R}$ such that $|f(x)| \le Mx^\alpha$. Then,*

$$\int_N^\infty \frac{f(x)}{x^{s+1}} dx$$

converges for $s \in \mathbb{C}$ with $\sigma > \alpha$ and defining $I(s)$ and the value of the integral, then $I(s)$ is analytic in Re $s > \alpha$ and its derivative is given by

$$I'(s) = -\int_N^\infty \frac{f(x)\log x}{x^{s+1}} dx.$$

Proof. The proof is similar to the one given to prove that Dirichlet Series are analytic in their half plane of convergence.

Note that

$$\left| \int_N^\infty \frac{f(x)}{x^{s+1}} dx \right| \le \int_N^\infty \frac{|f(x)|}{x^{\sigma+1}} dx \le \int_N^\infty \frac{Mx^\alpha}{x^{\sigma+1}} dx = M \int_N^\infty \frac{1}{x^{1+(\sigma-\alpha)}} dx.$$

Therefore, for $s \in \mathbb{C}$ with $\sigma > \alpha$ the integral converges since $1 + (\sigma - \alpha) > 1$.

To prove that $I(s)$ is analytic, consider the sequence of functions $\{I_n(s)\}_{n \in \mathbb{N}}$ defined by

$$I_n(s) = \int_N^{N+n} \frac{f(x)}{x^{s+1}} dx.$$

By theorem D.2, each $I_n(s)$ is analytic in the whole of $\mathbb{C}$, moreover,

$$I_n'(s) = -\int_N^{N+n} \frac{f(x)\log x}{x^{s+1}} dx.$$

It will be shown that the sequence $\{I_n(s)\}_{n \in \mathbb{N}}$ converges uniformly on every compact subset of Re $s > \alpha$ from which will follow that $I(s)$ is analytic in this region and that $I_n'(s) \to I'(s)$ when $n \to \infty$.

Take a compact subset K in Re $s > \alpha$ and let $a \in \mathbb{R}$ be such that for every $s = \sigma + it \in K$ one has $\alpha < a < \sigma$. If $s \in K$, then

$$\begin{aligned} |I(s) - I_n(s)| &= \left| \int_{N+n}^\infty \frac{f(x)}{x^{s+1}} dx \right| \le \int_{N+n}^\infty \frac{Mx^\alpha}{x^{\sigma+1}} dx \\ &= \frac{M}{(\sigma-\alpha)(N+n)^{\sigma-\alpha}} \le \frac{M}{(a-\alpha)(N+n)^{a-\alpha}} \end{aligned}$$

where the last inequality comes from the fact that $\sigma - \alpha > a - \alpha$. Since the last term tends to zero when $n \to \infty$ independently of s, uniform convergence on K is proved. ∎

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