

Exam 3, Math 129, Summer 1 2008

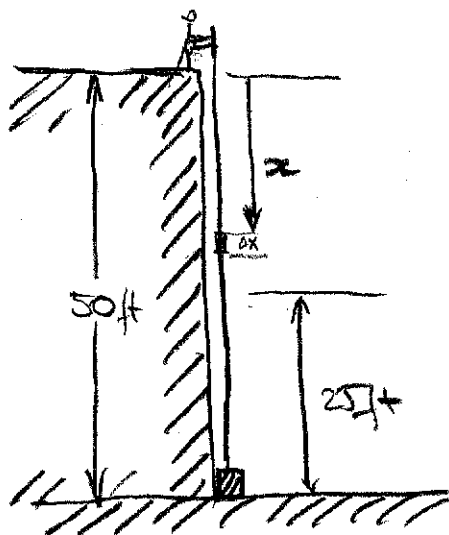
Solutions

Math 129, Section 2

~~Show Your Work!~~

Page 1 of 8

1. (10) A worker is on a roof of a 50 ft tall building. He needs to lift a 100 lb bucket of cement from the ground to a point 25 ft above the ground by pulling on a rope weighing 1 lb/ft. How much work is required? Carefully define your variable of integration.



Method 1: Rope is "cut" in little pieces, each piece being lifted a variable distance.

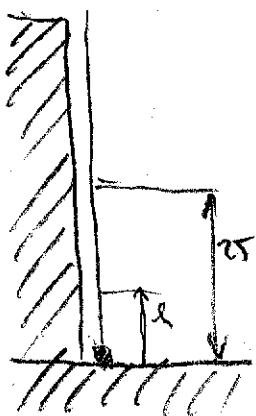
Let x denote the distance to the top of the building.

$$\begin{aligned}
 \text{Work} &= \text{work on bucket} + \left(\text{work on the first } 25 \text{ ft of rope} \right) + \left(\text{work on the top } 25 \text{ ft of rope} \right) \\
 &= \underbrace{100 \cdot 25}_{\substack{\text{weight} \\ \text{distance lifted}}} + \underbrace{25 \cdot 25}_{\substack{\text{weight} \\ \text{distance lifted}}} + \int_0^{25} \underbrace{1 \cdot x}_{\substack{\text{distance} \\ \text{weight of a little piece of rope}}} dx \\
 &= \frac{6875}{2} \text{ ft}\cdot\text{lb}
 \end{aligned}$$

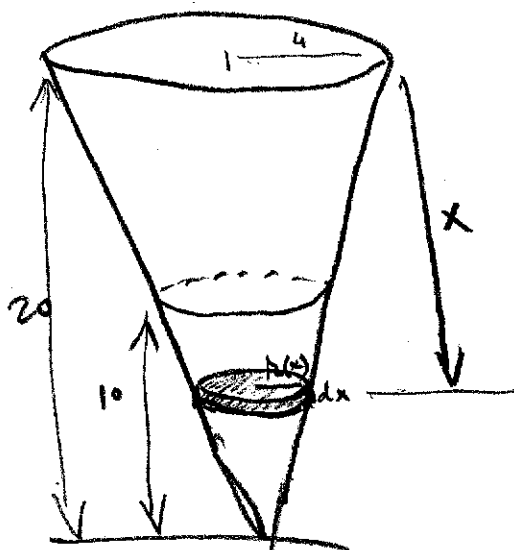
Method 2 (by Margaret Preston) Variable force, as in the example with the Hooke's Law. Let h be the height above the ground.

$$\text{Work} = \int_0^{25} \underbrace{(100 + 1 \cdot (50 - h))}_{\substack{\text{weight when } h \text{ ft of the rope} \\ \text{were lifted}}} \cdot \underbrace{dh}_{\text{distance}} = \frac{6875}{2}$$

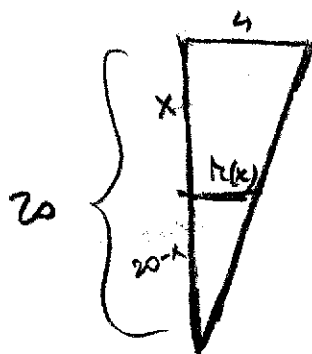
$$\boxed{\text{Work} = \frac{6875}{2} \text{ ft}\cdot\text{lb}}$$



2. (15) A water tank is in shape of right circular cone with height 20 ft and radius 4 ft at the top. Find the work required to pump all the water over the top of the tank, if the tank is filled with water to a depth of 10 ft. Recall that one cubic foot of water weighs 62.4 lb. Carefully define your variable of integration.



Let x be the distance to the top of the tank. We decompose the volume of water in horizontal slices. These are cylinders of height Δx and radius $r(x)$. We obtain $r(x)$ from a similar triangle.



$$\frac{r(x)}{4} = \frac{20-x}{20}$$

$$r(x) = \frac{20-x}{5}$$

$$\text{Work} = \sum \text{weight} \cdot \text{distance} = \int_{10}^{20} \underbrace{62.4}_{\text{density}} \cdot \underbrace{\pi \frac{1}{25}(20-x)^2}_{\text{volume of slice}} \cdot \underbrace{x}_{\text{distance}} dx$$

$$= \frac{62.4}{25} \pi \int_{10}^{20} [20(20-x)^2 - (20-x)^3] dx$$

$$= \frac{62.4}{25} \pi \left[-20 \frac{(20-x)^3}{3} + \frac{(20-x)^4}{4} \right] \Big|_{10}^{20} = 10,400 \pi \text{ ft-lb}$$

$$\boxed{\text{Work} = 10,400 \pi \text{ ft-lb}}$$

3. (10) According with several studies, aspirin reduces the risk of a heart attack. You decide to take one tablet a day, each tablet containing 350 milligram of aspirin. The body eliminates the aspirin such that at the end of 24 hours about 0.5% of the initial amount is still left in the body.

- (a) Let Q_n designate the quantity, in milligrams, of aspirin in the body right after the n^{th} tablet. Find Q_1 , Q_2 , Q_3 , Q_4 , and the general formula for Q_n .

Let $r = 0.005$. We have

$$Q_1 = 350$$

$$Q_2 = 0.005 Q_1 + 350 = (r+1) \cdot 350$$

$$Q_3 = 0.005 \cdot Q_2 + 350 = (r^2 + r + 1) \cdot 350$$

$$Q_4 = 0.005 \cdot Q_3 + 350 = (r^3 + r^2 + r + 1) \cdot 350$$

$$Q_n = (r^{n-1} + \dots + r + 1) \cdot 350 = \left(\frac{1-r^n}{1-r} \right) \cdot 350$$

- (b) What is $\lim_{n \rightarrow \infty} Q_n$?

$$\lim_{n \rightarrow \infty} Q_n = \frac{1}{1-r} 350 = 351.759$$

- (c) Let P_n designate the quantity, in milligrams, of aspirin in the body at the end of the n^{th} day, right before the next tablet is taken. Find P_1 , P_2 , P_3 , P_4 , and the general formula for P_n .

$$P_1 = 0.005 \cdot 350 = r \cdot 350$$

$$P_2 = 0.005 \cdot Q_2 = r(r+1) 350$$

$$P_3 = 0.005 \cdot Q_3 = r(r^2 + r + 1) 350$$

$$P_4 = 0.005 \cdot Q_4 = r(r^3 + r^2 + r + 1) 350$$

$$P_n = r \cdot \frac{1-r^n}{1-r} \cdot 350$$

4. (16) Use the integral test to decide whether the series below converge or diverge. In the case of convergence, provide an upper bound for the sum.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2} + n^{1/2}}$

Method 1 (by Kane Peterson) Apply directly the integral test:

$$\int_1^{\infty} \frac{1}{\sqrt{x}(1+x)} dx = 2 \arctan(\sqrt{x}) \Big|_1^{\infty} = 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \pi/2$$

So the series converges with upper bound $\pi/2$.

Method 2: We first use comparison $\frac{1}{n^{3/2} + n^{1/2}} < \frac{1}{n^{3/2}}$, then integral test:

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{3/2}} = 1 + \int_1^{\infty} \frac{1}{x^{3/2}} dx = 1 - 2 \frac{1}{x^{1/2}} \Big|_1^{\infty} = 3$$

Convergent ☒ Divergent ☐ Bound = $\pi/2$

(b) $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$

Because $\int_2^{\infty} \frac{1}{x \ln x} dx = \ln(\ln x) \Big|_2^{\infty} = \infty$

the series itself is divergent

Convergent ☐ Divergent ☒ Bound =

5. (10) Is the series

$$\overbrace{\frac{a_2}{2}} - \overbrace{\frac{a_3}{3}} + \overbrace{\frac{a_4}{4}} - \overbrace{\frac{a_5}{5}} + \frac{6}{35} - \frac{7}{48} + \dots$$

absolutely convergent, conditionally convergent, or divergent? Explain your answer.

Solution: The series is alternating $a_2 - a_3 + a_4 - \dots = \sum_{n=2}^{\infty} (-1)^n a_n$,
with $a_n = \frac{n}{n^2-1}$

We have : • $a_n > a_{n+1} \Leftrightarrow \frac{n}{n^2-1} > \frac{n+1}{(n+1)^2-1} \Leftrightarrow n(n^2+2n) > (n^2-1)(n+1)$
 $\Leftrightarrow \cancel{n^3} + 2n^2 > \cancel{n^3} + n^2 - n - 1$
 $\Leftrightarrow n^2 + n + 1 > 0$

•• $\lim_{n \rightarrow \infty} a_n = 0$

Consequently the alternating series test applies and the given series is convergent.

For the absolute values series, $\sum_{n=2}^{\infty} \frac{n}{n^2-1}$ we apply
The integral test

$$\int_2^{\infty} \frac{x}{x^2-1} dx = \frac{1}{2} \ln(x^2-1) \Big|_2^{\infty} = \infty$$

We conclude that the absolute values series diverges.

Absolutely convergent _____ Conditionally convergent ☒ Divergent _____

6. (15) Determine whether the series below converge or diverge. State which convergence test you used.

$$(a) \sum_{n=4}^{\infty} \frac{\sqrt{n-3}}{n+1}$$

$$\text{Let } a_n = \frac{\sqrt{n-3}}{n+1} \quad \text{and } d_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{d_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n-3} \cdot \sqrt{n}}{n+1} = \lim_{n \rightarrow \infty} \sqrt{\frac{n(n-3)}{(n+1)^2}} = 1$$

The limit comparison test applies.

Convergent _____ Divergent ☒ Test limit comparison

$$(b) \sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$$

$$\text{Let } a_n = \cos\left(\frac{1}{n}\right)$$

$$\lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1$$

The general term does not go to zero, so series diverges.

Convergent _____ Divergent ☒ Test "go to 0 test"

$$\begin{aligned} (c) \sum_{n=1}^{\infty} \frac{\pi e^n - \sqrt{2}}{4^n} &= \pi \sum_{n=1}^{\infty} \left(\frac{e}{4}\right)^n - \sqrt{2} \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n \\ &= \pi \cdot \frac{\frac{e}{4}}{1 - \frac{e}{4}} - \sqrt{2} \frac{\frac{1}{4}}{1 - \frac{1}{4}} \end{aligned}$$

Convergent ☒ Divergent _____ Test algebraic manipulations + geometric series test

7. (16) Find the radius of convergence and the interval of convergence for the series given below. Do check the ends of the interval as well.

$$(a) \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} (x - \pi)^{2k}$$

We consider the series as $\sum_{k=0}^{\infty} a_k$, where $a_k = (-1)^k \frac{1}{(2k)!} (x - \pi)^{2k}$

$$\left| \frac{a_{k+1}}{a_k} \right| = \frac{1}{(2k+2)!} (x - \pi)^{2k+2} \cdot \frac{(2k)!}{(x - \pi)^{2k}} = \frac{(x - \pi)^2}{(2k+1)(2k+2)} \xrightarrow{k \rightarrow \infty} 0 < 1 \text{ for all } x$$

Consequently, the ratio test implies that $R = \infty$

$$R = \infty \quad \text{Interval of convergence} = (-\infty, \infty)$$

$$(b) (x+1) + \frac{1}{2}(x+1)^2 + \frac{1}{3}(x+1)^3 + \frac{1}{4}(x+1)^4 + \dots$$

Here the series is $\sum_{n=1}^{\infty} C_n (x+1)^n$, where $C_n = \frac{1}{n}$

$$R = \frac{1}{\lim_{n \rightarrow \infty} \left| \frac{C_{n+1}}{C_n} \right|} = \frac{1}{\lim_{n \rightarrow \infty} \frac{n}{n+1}} = 1$$

At $x=0$, series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$, so divergent (harmonic)

At $x=-2$, series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, so convergent (oscillating harmonic)

$$R = 1 \quad \text{Interval of convergence} = [-2, 0)$$

8. (8) Suppose that the power series $\sum_{n=0}^{\infty} C_n(x-1)^n$ converges for $x = 3$ and diverges for $x = -2$. Which of the following are True (T), False (F), or impossible (I) to determine. Briefly justify your answer.

The key is the next picture:

- (a) T ☒ F ☐ I The power series diverges for all $x < 0$.

For all x in $(-1, 0)$ we are guaranteed convergence.

- (b) T ☐ F ☒ I The power series converges for $x = 3.5$.

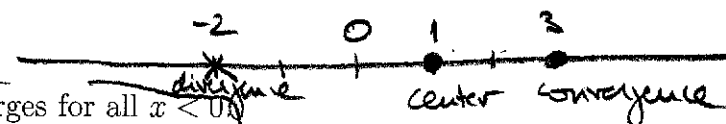
3.5 is in the "gray area".
We do not have enough information to decide what R is, so knowing that $2 \leq R \leq 3$ does not help answering this.

- (c) ☒ T ☐ F ☐ I The power series converges for $x = 0$.

0 belongs to $(-1, 3)$

- (d) ☒ T ☐ F ☐ I The power series diverge for all $x > 4$.

This is guaranteed divergence.



This means that the radius of convergence must satisfy

$$2 \leq R \leq 3$$

with guaranteed convergence in $(-1, 3)$ and guaranteed divergence outside $[-2, 4]$.

All these follow from the properties of the convergence of power series.