Exam 1, Math 129, Summer I 2008

Solutions

1. (15)

(a) (8 pts) Find $I = \int_0^{\pi^2/4} \frac{\cos(\sqrt{x})}{\sqrt{x}}\, dx.$ $= 2\sin(\sqrt{x})\Big|_0^{\pi^2/4} = 2$

$w = \sqrt{x}$

$dw = \frac{1}{2}\frac{1}{\sqrt{x}}dx$

$\boxed{I = 2}$

(b) (7 pts) Find $\int (3t+1)\sqrt[3]{2t-1}\, dt.$ $= \frac{1}{2}\int \left(3\cdot\frac{w+1}{2} + 1\right) w^{1/3}\, dw$

$w = 2t-1$

$dw = 2dt$

$= \frac{1}{4}\int \left(3w^{4/3} + 5w^{1/3}\right) dw$

$= \frac{1}{4}\left(\frac{9}{7} w^{7/3} + \frac{15}{4} w^{4/3}\right) + C$

Answer: $\boxed{\frac{9}{28}(2t-1)^{7/3} + \frac{15}{16}(2t-1)^{4/3} + C}$

2. (10) Let h be a twice differentiable function with $h(0) = -1$, $h(2) = \pi$, and $h'(2) = 3$. Find $\int_0^2 x\, h''(x)\, dx$.

Solution: This is an application of integration by parts:
Let $u(x) = x$ and $v'(x) = h''(x)$, such that $v(x) = h'(x)$,
and we have:

$$\int_0^2 x\, h''(x) = x\, h'(x)\Big|_0^2 - \int_0^2 h'(x)$$
$$= \left[x\, h'(x) - h(x)\right]\Big|_0^2$$
$$= 2 \cdot h'(2) - h(2) + h(0) = 5 - \pi$$

$\int_0^2 x\, h''(x)\, dx =$ $\boxed{5-\pi}$

3. (10) Evaluate the following integral (you can use the table of integrals): $I = \int \frac{3}{9y^2 - 4}\, dy$.

Solution: $I = \frac{1}{3} \int \frac{1}{y^2 - \frac{4}{9}}\, dy = \frac{1}{3} \int \frac{1}{(y-\frac{2}{3})(y+\frac{2}{3})}\, dy$

$$= \cancel{\tfrac{1}{3}}\, \frac{\cancel{3}}{4} \cdot \int \left(\frac{1}{y - 2/3} - \frac{1}{y + 2/3}\right) dy$$

$$= \boxed{\frac{1}{4}\left(\ln\left|y - \tfrac{2}{3}\right| - \ln\left|y + \tfrac{2}{3}\right|\right) + C}$$

$I =$ ____

4. (10) The voltage, V, in an electrical circuit is given as a function of time, t, by $V = V_0 \cos(\omega t + \varphi)$, where V_0, ω, and φ are positive constants.

(a) What is the average value of V^2 over one period? (Recall that the length of a period is $T = 2\pi/\omega$, and that the average of a function f over an interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.)

Solution: Average $= \dfrac{1}{\frac{2\pi}{\omega}} \displaystyle\int_0^{2\pi/\omega} V_0^2 \cdot \cos^2(\omega t + \varphi)\,dt$

$$= \frac{V_0^2 \cdot \omega}{2\pi} \int_0^{2\pi/\omega} \frac{1 + \cos(2(\omega t + \varphi))}{2}\,dt$$

$$= \frac{V_0^2 \omega}{4\pi} \cdot \left[t + \frac{1}{2\omega} \sin(2(\omega t + \varphi)) \right] \Big|_0^{2\pi/\omega}$$

$$= \frac{V_0^2 \cdot \omega}{4\pi} \cdot \frac{2\pi}{\omega}$$

$$= \boxed{\frac{V_0^2}{2}}$$

(b) Assume that the frequency ω is increased. What is the effect of this increase on the average value of V^2 that you computed in part (a)?

No effect: the average does not depend on ω.

5. (10) Use the method of partial fractions to compute:

$$\int \frac{-2x^3 - x - 1}{x^3 + x}\, dx.$$

Solution

First we write $\dfrac{-2x^3-x-1}{x^3+x} = -2 + \dfrac{x-1}{x^3+1}$.

Then: $\dfrac{x-1}{x^3-x} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} \implies$ $A=-1$, $B=1$, $C=-1$

So the integral is: $\int \left(-2 - \frac{1}{x} + \frac{\cancel{x}}{x^2+1} + \frac{1}{x^2+1}\right) dx =$

$$= \boxed{-2x - \ln|x| + \frac{1}{2}\ln(x^2+1) + \arctan x + C}$$

Answer: ______________

6. (10) Use a trigonometric substitution to evaluate $I = \displaystyle\int_0^1 \frac{1}{(x^2+4)^{3/2}}\, dx.$

Solution: Let $x = 2\dfrac{\sin\theta}{\cos\theta} = 2\tan\theta$

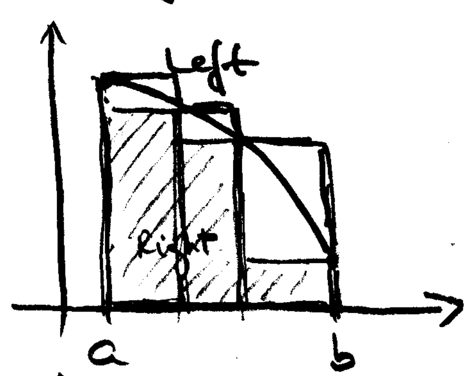
$dx = \dfrac{2}{\cos^2\theta}\, d\theta$

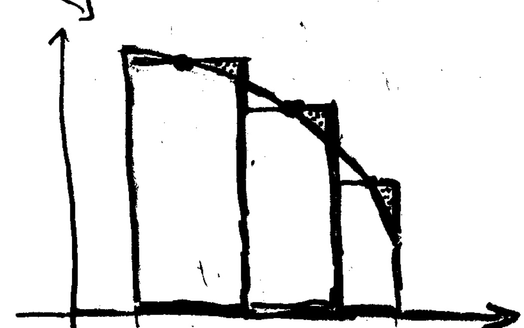
$$I = \frac{1}{8}2\int_0^{\arctan(1/2)} \frac{1}{\frac{1}{\cos^3\theta}} \cdot \frac{1}{\cos^2\theta}\, d\theta = \frac{1}{4}\sin\theta \Big|_0^{\arctan 1/2}$$

$$\boxed{I = \frac{1}{4}\sin(\arctan 1/2)}$$

7. (16) Consider a function f that is strictly decreasing and concave down on an interval $[a, b]$. Indicate in the table below, for each of the Riemann sums, whether it represents an *underestimate* or an *overestimate* of $\int_a^b f(x)\,dx$. Briefly justify your answers.

Left(n)	**Right(n)**	**Mid(n)**	**Trap(n)**
overestimate	underestimate	overestimate	underestimate





we have, for decreasing functions:

$$\text{Left}(n) \geq \int_a^b f(x)\,dx \geq \text{Right}(n)$$

we have, for concave down functions:

$$\text{Trap}(n) \leq \int_a^b f(x)\,dx \leq \text{Mid}(n)$$

8. (7) Someone scrambled the columns with errors corresponding to Right(n), Mid(n), and Simp(n) of a function f over an interval $[a, b]$. Can you determine which is which? Fill in the top of the table and briefly justify your answers.

n	Errors in ? Simpson rule	Errors in ? Right rule	Errors in ? Mid rule
10	0.0000008000	-0.0360448725	-0.0007263637
100	0.0000000001	-0.0037354170	-0.0000072914
1000	0.0000000000	-0.0003748542	-0.0000000729
10000	0.0000000000	-0.0000374985	-0.0000000007

We know that increasing the number of subdivisions by 10 makes the error smaller by a factor of

- 10 in left/right rules,
- 10^2 in trap/mid point rules, and
- 10^4 in Simpson's rule.

9. (12) For the improper integrals below decide whether they are convergent or divergent. Circle **C**onvergent or **D**ivergent. Briefly justify your answer.

(a) C D $\int_2^\infty \frac{1}{(x-1)^2}\,dx \underset{w=x-1}{=} \int_1^\infty \frac{1}{w^2}\,dw$ = convergent $(p>1)$

(b) C D $\int_{-\infty}^\infty \frac{1}{x^2+1}\,dx = \arctan x\Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

convergent

(c) C D $\int_{-\infty}^0 \frac{e^x}{1+e^x}\,dx = \int_1^2 \frac{1}{w}\,dw = \ln(2)$

$w = 1+e^x$
$dw = e^x dx$

convergent

(d) C D $\int_{\sqrt{2}}^\infty \frac{1}{\sqrt{x^2-1}}\,dx \underset{\text{table}}{\overset{\#29}{=}} \lim_{A\to\infty} \ln\left(x+\sqrt{x^2-1}\right)\Big|_{\sqrt{2}}^{\infty} = \infty$

divergent