

1 The Axiom of Extensionality

Primitive notion: Set

A set is a group, a collection, or an aggregate of things. In fact, the words “set”, “group”, “collection”, and “aggregate” are all synonyms denoting the same primitive notion. A set need not be a collection of things that are physically grouped; rather it is in thought that things form a set. For example, there may be a set of a fish and the moon. Use of the word “set” shall be preferred over its synonyms. Hence, the study of sets is named set theory.

All statements in set theory involve sets. As such, the phrase “is a set” could unnecessarily lengthen sentences. To shorten notation, the phrase “is a set” will be avoided by using uppercase letters to denote sets. For example, we have the following definitions.

$\forall A, \phi(A)$ will denote $\forall a, (a \text{ is a set} \Rightarrow \phi(a))$. The statement $\forall A, \phi(A)$ is read “For any set A , $\phi(A)$ ”.
 $\exists B, \phi(B)$ will denote $\exists b, (b \text{ is a set} \wedge \phi(b))$. The statement $\exists B, \phi(B)$ is read “There exists a set B such that $\phi(B)$ ”.

The use of the quantifiers \forall and \exists in the previous definitions is indicative of the statements of quantifier logic.

Primitive notion: Element

An element of a set is a thing contained in a set. For example, if there is a set of a fish and the moon, then the moon is an element of that set. The primitive notion “element” can also be used to express the statement “ x is an element of y ” even if y is not a set, but then the statement is false. The notation $x \in y$ denotes that a thing x is an element of a thing y , and the notation $x \notin y$ denotes that a thing x is *not* an element of a thing y .

Definition: $\forall x, \forall y, (x \in y := x \text{ is an element of } y)$

Definition: $\forall x, \forall y, (x \notin y := \neg(x \in y))$

A thing which is not a set is called an individual. As mentioned above, if x is an individual, then y is not an element of x for any y . This is called the axiom of individuals.

Definition: $\forall x, (x \text{ is an individual} := \neg(x \text{ is a set}))$

The Axiom of Individuals: $\forall x, (x \text{ is an individual} \Rightarrow (\forall y, y \notin x))$

Unlike individuals which have no elements, a set is completely characterized by its elements. Two sets that have the exact same elements are equal. This axiom is called the axiom of extensionality.¹

The Axiom of Extensionality: $\forall A, \forall B, (\forall x, (x \in A \Leftrightarrow x \in B) \Rightarrow A = B)$

The axiom of extensionality implies several statements about subsets and supersets. A set A is a subset of a set B , if A is included in B . That is, if every element of A is an element of B . The

¹Extensionality means determining objects to be equal by an external property and not their formulation

notation $A \subseteq B$ denotes that A is a subset of B . A set A is a superset of a set B , if A includes B . That is, if every element of B is an element of A . The notation $A \supseteq B$ denotes that A is a superset of B . Formally, subsets and supersets are defined as follows.

Definition: $\forall A, \forall B, (A \subseteq B := \forall x, (x \in A \Rightarrow x \in B))$

Definition: $\forall A, \forall B, (A \supseteq B := \forall x, (x \in A \Leftarrow x \in B))$

Next are five minor propositions about subsets and supersets to acquaint you with the terms. None of their proofs involve the axiom of extensionality. Proposition 1.1 and proposition 1.2 state that every set is a subset and a superset of itself. Proposition 1.3 states that a set A is a subset of a set B if and only if B is a superset of A . Proposition 1.4 states that if a set A is a subset of a set B and B is a subset of a set C , then A is a subset of C . Proposition 1.5 states that if a set A is a superset of a set B and B is a superset of a set C , then A is a superset of C .

Proposition 1.1: $\forall A, A \subseteq A$

Proof: $\forall x, (x \in A \Rightarrow x \in A)$. Therefore, $A \subseteq A$. QED

Proposition 1.2: $\forall A, A \supseteq A$

Proof: $\forall x, (x \in A \Leftarrow x \in A)$. Therefore, $A \supseteq A$. QED

Proposition 1.3: $\forall A, \forall B, (A \subseteq B \Leftrightarrow B \supseteq A)$

Proof: $A \subseteq B \Leftrightarrow \forall x, (x \in A \Rightarrow x \in B) \Leftrightarrow \forall x, (x \in B \Leftarrow x \in A) \Leftrightarrow B \supseteq A$ QED

Proposition 1.4: $\forall A, \forall B, \forall C, ((A \subseteq B \wedge B \subseteq C) \Rightarrow A \subseteq C)$

Proof:

$$\begin{aligned} (A \subseteq B \wedge B \subseteq C) &\Rightarrow \forall x, (x \in A \Rightarrow x \in B) \wedge \forall x, (x \in B \Rightarrow x \in C) \\ &\Rightarrow \forall x, ((x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in C)) \\ &\Rightarrow \forall x, (x \in A \Rightarrow x \in C) \\ &\Rightarrow A \subseteq C \end{aligned}$$

QED

Proposition 1.5: $\forall A, \forall B, \forall C, ((A \supseteq B \wedge B \supseteq C) \Rightarrow A \supseteq C)$

Proof:

$$\begin{aligned} (A \supseteq B \wedge B \supseteq C) &\Rightarrow \forall x, (x \in A \Leftarrow x \in B) \wedge \forall x, (x \in B \Leftarrow x \in C) \\ &\Rightarrow \forall x, ((x \in A \Leftarrow x \in B) \wedge (x \in B \Leftarrow x \in C)) \\ &\Rightarrow \forall x, (x \in A \Leftarrow x \in C) \\ &\Rightarrow A \supseteq C \end{aligned}$$

QED

The following theorem, the theorem of set equality, states that if A is both a subset and a superset of B , then the two sets are equal. Proposition 1.3 equates this to showing that if two sets are subsets of each other, then they are equal; and that if two sets are supersets of each other, then they are equal. When it is difficult to prove directly that two sets are equal, it is often easier to prove the two sets contain each other, and then use the theorem of set equality.

Theorem of Set Equality: $\forall A, \forall B, ((A \subseteq B \wedge A \supseteq B) \Rightarrow A = B)$

Proof: $(A \subseteq B \wedge A \supseteq B) \Rightarrow \forall x, (x \in A \Rightarrow x \in B) \wedge \forall x, (x \in A \Leftarrow x \in B)$
 $\Rightarrow \forall x, ((x \in A \Rightarrow x \in B) \wedge (x \in A \Leftarrow x \in B))$
 $\Rightarrow \forall x, (x \in A \Leftrightarrow x \in B)$
 $\Rightarrow A = B$, by the axiom of extensionality. QED

The theorem of set equality can be used to prove that if there is a set satisfying some property which intuitively is the smallest such set, then it is in fact the smallest such set.

Theorem of Unique Smallest Sets:

$\exists A, (\phi(A) \wedge \forall B, (\phi(B) \Rightarrow A \subseteq B)) \Rightarrow \exists! A, (\phi(A) \wedge \forall B, (\phi(B) \Rightarrow A \subseteq B))$

Proof: Suppose there exists a set A such that $\phi(A)$ and $\forall B, (\phi(B) \Rightarrow A \subseteq B)$. Let A' be another such set. Then $\phi(A')$ and $\phi(A') \Rightarrow A \subseteq A'$. So $A \subseteq A'$. Likewise, $\phi(A)$ and $\phi(A) \Rightarrow A' \subseteq A$. So $A' \subseteq A$. By the theorem of set equality, $A = A'$. Therefore, A is unique set such that $\phi(A) \wedge \forall B, (\phi(B) \Rightarrow A \subseteq B)$. QED

Theorem of Unique Largest Sets:

$\exists A, (\phi(A) \wedge \forall B, (\phi(B) \Rightarrow A \supseteq B)) \Rightarrow \exists! A, (\phi(A) \wedge \forall B, (\phi(B) \Rightarrow A \supseteq B))$

Proof: Suppose there exists a set A such that $\phi(A)$ and $\forall B, (\phi(B) \Rightarrow A \supseteq B)$. Let A' be another such set. Then $\phi(A')$ and $\phi(A') \Rightarrow A \supseteq A'$. So $A \supseteq A'$. Likewise, $\phi(A)$ and $\phi(A) \Rightarrow A' \supseteq A$. So $A' \supseteq A$. By the theorem of set equality, $A = A'$. Therefore, A is the unique set such that $\phi(A) \wedge \forall B, (\phi(B) \Rightarrow A \supseteq B)$. QED

The theorem of unique smallest sets is more useful than the theorem of unique largest sets, because it will often be easy to produce such largest set. The theorem of unique smallest sets will be used to consider sets generated by a certain condition.

2 Russel's Paradox and the Axiom Schema of Separation

The axiom of extensionality says that the essence of a set is its elements. This axiom, however, is irrelevant until an axiom is proposed that implies the existence of at least one set. Initially, I accepted intuitively one such axiom: the axiom schema of abstraction. It says that for every property there is a set containing precisely those elements that satisfy the property. Formally, the

property is introduced by a statement ϕ and a thing x satisfies that property if and only if $\phi(x)$. In mathematical notation, the axiom schema of abstraction is written as follows:

The Axiom Schema of Abstraction: $\forall^s \phi, \exists A, \forall x, (x \in A \Leftrightarrow \phi(x))$

Unfortunately, it's false. Even still the axiom schema of abstraction has the word axiom in its name because it was originally proposed as an axiom of mathematics. Of course, it is no longer accepted as an axiom of mathematics. Below is a proof, known as Russel's paradox, that the axiom schema of abstraction is false.

Proposition 2.1: *The axiom schema of abstraction is false.*

Proof (Russel's Paradox): Consider the property of a thing not being an element of itself; formally, let $\phi(x)$ be the statement $x \notin x$. Then supposing the axiom schema of abstraction is true, there is a set A of all things that are not elements of themselves. That is, $\exists A, \forall x, (x \in A \Leftrightarrow x \notin x)$. Next consider whether or A is an element of itself.

- If $A \notin A$, then A satisfies the property of not being an element of itself. Since A is a set containing all things that are not elements of themselves, $A \in A$. So $A \notin A \Rightarrow A \in A$.
- If $A \in A$, then A is not an element of itself, because every element of A is not an element of itself. So $A \in A \Rightarrow A \notin A$.

Therefore, $A \in A \Leftrightarrow A \notin A$, which is a logical contradiction. This could be obtained more succinctly noting that $x \in A \Leftrightarrow x \notin x$ for all x . In particular, $x \in A \Leftrightarrow x \notin x$ holds for $x = A$. That is, $A \in A \Leftrightarrow A \notin A$, which is a logical contradiction. Therefore, the axiom schema of abstraction is false. QED

Before learning Russel's paradox, I accepted intuitively the axiom schema of abstraction, because I did not consider the possibility of sets of sets, only sets of individuals. For the purpose of an analogy, let velvet bags represent sets and consider a pile of marbles each of which represent an individual. I first viewed the axiom schema of separation as saying that one can take all of the marbles of a certain type from the pile and put them in one bag. While the axiom schema of abstraction implies this, it says more. The axiom schema of abstraction also implies that one can take all of the bags of a certain type and them in one bag. Imagine then holding one bag and placing inside all the bags of a certain type bag only to find out that the bag you are holding is one of the bags you placed inside. Bizarre!

To show that a set exists, I will begin with the axiom of a universal set. That is, there exists a set of all individuals.

Axiom of a Universal Set: $\exists A, \forall x, (x \in A \Leftrightarrow x \text{ is an individual})$

Proposition 2.2: $\exists! A, \forall x, (x \in A \Leftrightarrow x \text{ is an individual})$

Proof: By the axiom of a universal set, there is a set A such that $\forall x, (x \in A \Leftrightarrow x \text{ is an individual})$. If A' is a set such that $\forall x, (x \in A' \Leftrightarrow x \text{ is an individual})$, then $x \in A \Leftrightarrow x \text{ is an individual} \Leftrightarrow x \in A'$

for all x . So $x \in A \Leftrightarrow x \in A'$ for all x . Thus, $A = A'$. Therefore, A is the unique set such that $\forall x, (x \in A \Leftrightarrow x \text{ is an individual})$. QED

Definition: $\exists! A, \forall x, (x \in A \Leftrightarrow x \text{ is an individual}) \Rightarrow (\mathbb{U} := A)$

Proposition 2.3: $\forall x, (x \in \mathbb{U} \Leftrightarrow x \text{ is an individual})$

Proof: By definition, \mathbb{U} is the unique set such that $\forall x, (x \in \mathbb{U} \Leftrightarrow x \text{ is an individual})$. QED

Now, however, we have the problem that we do not have many sets. For the purpose of mathematics we shall need to group things in many different ways. In particular, we shall need to form sets of sets. To remedy the problem of the axiom schema of abstraction, we only consider subsets. That is, for any set we can form a subset of the set consisting of all the elements that satisfy a given property. This is like the axiom schema of abstraction except that we cannot form a set of elements unless it is already contained in a larger set. That is, we have already verified that we can group the items we want to group along with others. This axiom is called the axiom schema of separation as we are separating the elements that will form our set.

Axiom Schema of Separation: $\forall^s \phi, \forall A, \exists B, \forall x, (x \in B \Leftrightarrow (x \in A \wedge \phi(x)))$

Here's how the axiom schema of separation avoids Russel's paradox. Let $\phi(x)$ be the statement $x \notin x$. Then by the newly proposed axiom, there exists a set A such that $\forall x, (x \in A \Leftrightarrow x \notin x \wedge \theta(x))$. In particular, $A \in A \Leftrightarrow (A \notin A \wedge \theta(A))$. That implies $A \notin A \wedge \neg\theta(A)$. $\neg\theta(A)$ is the important part of that statement. So long as $\neg\theta(A)$, the issue can be resolved by having $A \notin A$.

To be continued...